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# GÖRTLER VORTICES IN GROWING BOUNDARY LAYERS: THE LEADING EDGE RECEPTIVITY PROBLEM, LINEAR GROWTH AND THE NONLINEAR BREAKDOWN STAGE

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## ABSTRACT

Görtler vortices are thought to be the cause of transition in many fluid flows of practical importance. In this paper a review of the different stages of vortex growth is given. In the linear regime nonparallel effects completely govern this growth and parallel flow theories do not capture the essential features of the development of the vortices. A detailed comparison between the parallel and nonparallel theories is given and it is shown that at small vortex wavelengths the parallel flow theories have some validity; otherwise nonparallel effects are dominant. New results for the receptivity problem for Görtler vortices are given; in particular vortices induced by free-stream perturbations impinging on the leading edge of the wall are considered. It is found that the most dangerous mode of this type can be isolated and its neutral curve is determined. This curve agrees very closely with the available experimental data. A discussion of the different regimes of growth of nonlinear vortices is also given. Again it is shown that, unless the vortex wavelength is small, nonparallel effects are dominant. Some new results for nonlinear vortices of  $O(1)$  wavelengths are given and compared to experimental observations. The agreement between theory and experiment is shown to be excellent up to the point where unsteady effects become important. For small wavelength vortices the nonlinear regime is of particular interest since there a strongly nonlinear theory can be developed. Here the vortices can be large enough to drive the mean state which then adjusts itself to make all modes neutral. The breakdown of this nonlinear state into a three-dimensional time dependent flow is also discussed.

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## 1 Introduction

It is now some fifty years since H. Görtler published his 1940 paper on the centrifugal instability of a boundary layer on a concave wall. The mechanism discussed by Görtler is essentially the same as that investigated some twenty years earlier by G.I. Taylor, who was concerned with the instability of flows between concentric cylinders. In order to be consistent with standard practice we refer to the vortex instabilities discussed by these authors as Görtler and Taylor vortices respectively. However, there should be no misunderstanding about the relationship between the instabilities; they are caused by precisely the same centrifugal mechanism but, since the basic state in which Görtler vortices develop is spatially varying, it turns out that both the linear and nonlinear developments of the two types of vortices are quite distinct.

Whilst considerable effort has been made to understand the different stages in the development of Taylor vortices there has, in comparison, been little work done on Görtler vortices. In fact the Görtler mechanism is of significantly greater relevance to practical flow situations than is the Taylor mechanism. Thus in flows as diverse as those over turbine blades and in the human aorta, the curvature of the flow streamlines is sufficient to induce Görtler vortices. Another situation where Görtler vortices are thought to be the cause of transition to turbulence is the aerodynamic one. Thus the flow over the concave section of a Laminar Flow Wing or that in a jet engine inlet can support the mechanism. In Figure 1.1 we reproduce a picture of Görtler vortices in the concave section of a Laminar Flow Wing developed at NASA Langley; the picture was kindly supplied to the author by Dr. S. Mangalam. The vortices shown in the picture have been made visible by sublimating

chemicals and occur in the concave section of the wing. Furthermore, recent research on the later stages of transition in flat plate boundary layers has shown that interacting wave systems can play the role of streamline curvature and induce what are essentially Görtler vortex structures, see Hall and Smith(1988,1989a,b). Apart from the situations discussed above where Görtler vortices are known to be important there are other, less obvious, situations where they occur. Thus for example in the 1980 blast at Mount St. Helens longitudinal vortices set up in the flowing lava generated erosional furrows, Kieffer and Sturtevant(1988).

In the Taylor vortex problem remarkable progress has been made towards an understanding of the sequence of bifurcations which takes place when the speed of the inner cylinder is increased. There are two obvious reasons why little progress has been made in comparison with the Görtler problem. Firstly, the spatial development of the boundary layer in which the vortices grow means that a self-consistent asymptotic description of even the linear stages of the vortex development is difficult to obtain. Secondly, it is much easier to do careful experiments on the Taylor problem so that the theory has been to a great extent driven by the experiments. In comparison a Görtler vortex experiment is much more difficult to perform because of the inherent difficulties associated with flow quality control in an open system.

In fact the apparent similarities between Görtler and Taylor vortices are quite misleading, indeed little understanding of the growth of Görtler vortices can be obtained by studying the Taylor problem. Where appropriate in the remainder of this paper we shall highlight the major structural differences between the two vortex modes. Almost all of our

discussion will concern two-dimensional boundary layer flows but a restricted discussion of weakly three-dimensional boundary layer flows will be included. In the latter situation there is the possibility of yet another stationary vortex mode; we refer here to the so-called crossflow vortex instability whose structure was elucidated so clearly by Gregory, Stuart and Walker (1955). This mode is a Rayleigh instability of an effective velocity profile which has an inflection point at the critical layer and is thought to be the most likely cause of transition in flows over swept wings. In fact there is yet another stationary vortex instability possible in a three-dimensional boundary layer and this was described by Hall (1986). This disturbance is a Tollmien-Schlichting wave with an effective velocity profile having zero shear stress at the wall. We shall see later that the Görtler mechanism is destroyed by an asymptotically small spanwise mean flow so it appears likely that stationary vortex structures in three-dimensional boundary layers are associated with either Rayleigh or Tollmien-Schlichting waves.

At this stage it is appropriate for us to discuss the many experimental investigations of Görtler vortices that followed Görtler's original theoretical work. Perhaps the first researchers to observe Görtler vortices were Gregory and Walker (1956) who used the china-clay technique to visualize the vortices induced by protruberances in boundary layers. Earlier, Lieppmann (1943, 1945) had investigated the instability of boundary layers on curved walls and concluded that transition can be caused by Görtler vortices. Subsequently Aihara (1962) used dye to demonstrate the existence of vortices whilst Tani and Sakagami (1962) used smoke to visualize the disturbances. Wortmann (1964a,b) carried out more detailed flow visualization studies of Görtler vortices using the tellurium

method. Wortmann was able to show that in certain circumstances leaning vortices could be induced. The most detailed early observations of Görtler vortices were reported by Bippes(1972) and Bippes and Görtler(1972) who used the hydrogen bubble method to visualize vortices in water tunnels. These authors were also able to measure the eigenfunctions associated with the vortices since their experiments were performed on a relatively sharply curved wall.

More recently there have been several experiments designed to determine whether the Görtler mechanism is significant on the curved part of modern laminar flow wings , see for example Pfenninger et al (1980),Harvey and Pride (1982), Allison and Dagenhart(1987). In particular much work has been done at NASA Langley on a wing with a significant region of concave curvature on the underside of the wing just beyond the leading edge . In Figure 1.1 the longitudinal vortices in the concave region are clearly seen. Figure 1.2 shows a sketch of the flow pattern associated with the vortex flow. The Langley experiments were performed using sublimating chemicals to visualize the vortices and laser velocimetry to measure the disturbance velocity field. For more details of the NASA Langley experiments on Görtler vortices in boundary layers on laminar airfoils the reader is referred to Mangalam et al (1985,1987).

A question of some importance is that of how the upstream conditions in a Görtler vortex experiment influence the nature of the induced vortex system. This matter was addressed in the paper by Swearingen and Blackwelder(1983) who performed Görtler vortex experiments in a lowspeed wind tunnel with the aim of finding the mechanism which fixes the vortex wavelength. Earlier, Tani(1962) and Tani and Sakagami(1962) , along with



Bippes (1972) , had concluded from their experiments that the induced vortex wavenumber is independent of the free-stream speed, the spanwise dimension of the test section , and the streamwise location of the onset of vortex activity. Bippes found that the vortex wavenumber was however dependent on the nature of the incoming disturbance field. Swearingen and Blackwelder found that the observed wavelength of the vortices in their experiments could be altered by the introduction of strips of tape at the wall or by placing cylinders in the incoming flow. Bippes & Swearingen and Blackwelder attempted to find a mechanism which would cause the most amplified linear mode predicted by parallel flow theory to be observed. This they achieved by suitably positioning the screens ahead of the test section in their experiments. In fact there is no such thing as the most amplified linear mode since we shall see in the next section that nonparallel effects make the concept of a unique growth rate not tenable for the Görtler problem; this should not be interpreted as criticism of the latter experiments. Indeed, in view of the results discussed in the next section, we believe that the fact that these authors were able to induce a particular vortex wavelength shows conclusively that it is the upstream conditions which crucially select the wavelength and neutral position of the induced vortex. Moreover, Kottke(1986) , who was interested in determining the effects of Görtler vortices on heat transfer , found that in his experiment Görtler vortices could not even be observed unless a grid was placed in front of his test section. We note in passing here that the effect on heat transfer of Görtler vortices needs to be understood if efficient turbine blades are to be designed , see Finnis and Brown(1986).

The experimental papers discussed above were primarily concerned with demonstrating that Görtler vortices predicted by linear stability theory can be set up experimentally. On the basis of what is known about Taylor vortices one might expect that nonlinear effects will inhibit the growth predicted by linear theory and lead to finite amplitude equilibration. This was confirmed by the above experiments since it was found that after the onset of instability the vortex activity increased slowly in the streamwise direction. The absence of any kind of threshold amplitude response by a boundary layer to longitudinal vortices also suggests that nonlinear effects are indeed stabilizing in the Görtler problem.

However in the Taylor problem it is well-known that, when the Taylor number is increased sufficiently, the finite amplitude axisymmetric vortex system set up when the linear critical Taylor number is exceeded becomes unstable to a time-dependent three dimensional mode often referred to as a wavy vortex mode. After the onset of this instability the vortex boundaries have a wave superimposed on them and this wave travels in the azimuthal direction. In fact this secondary instability is merely a non-axisymmetric Taylor vortex destabilized by the initial finite amplitude state. A convincing theoretical description of the breakdown process leading to the wavy vortex state was given by Davey, DiPrima and Stuart(1969). Thus it would be surprising if such a process was not operational in the Görtler problem when the vortices develop downstream. We note that moving downstream in the Görtler problem roughly corresponds to increasing the Taylor number in the concentric cylinder problem.

At this stage it should be remembered out that Tollmien-Schlichting waves are another possible source of instability in a boundary layer whereas in Couette flow only centrifugal

modes are possible. It appears from the available experiments that the breakdown route for Görtler vortices is fixed by the size of the wall curvature. Not surprisingly for relatively large wall curvatures the Reynolds number is not large enough for Tollmien-Schlichting waves to be unstable and the wavy vortex mode is operational; see Bippes(1972), Aihara(1961) and Peerhossaini and Wesfreid(1988). In fact the wavy mode is apparently the cause of breakdown for the laminar flow wing case, Kohama(1987). Furthermore Kohama and Peerhossaini & Wesfreid identified two possible types of wavy vortex mode. The two modes respectively lead to oscillations of the cell boundaries at the top and bottom of the region of vortex activity. The characteristic frequency of the oscillation of the top boundary was found to be greatest but still not as large as a typical Tollmien-Schlichting frequency. We shall give a theoretical explanation for the existence of these distinct modes when we discuss theoretical work on nonlinear effects. At smaller values of the wall curvature Tollmien-Schlichting waves are involved in the secondary instability of the stationary vortex and the transition process has many similarities with flat plate transition. Some aspects of this interaction problem are now understood and a brief description of that work will be given in the conclusion to this paper.

Consider then the instability of the boundary layer flow

$$\underline{u} = \underline{u}_B = U_0(\bar{u}(x, y), Re^{-\frac{1}{2}}\bar{v}(x, y), 0), \quad (1.1)$$

to a spanwise periodic perturbation of wavelength comparable with the boundary layer thickness. Here  $x$  and  $y$  represent nondimensional distance along and normal to a wall of variable curvature  $\frac{1}{A}\chi(x)$  and  $x, y$  have been scaled on  $L$  and  $Re^{-\frac{1}{2}}L$  where  $L$  is a typical streamwise length and  $Re$  is the Reynolds number based on the length  $L$  and the

free stream speed  $U_0$ . We restrict our attention to walls of small curvature so the  $\frac{L}{A} \ll 1$ ; more precisely we note that the Görtler instability occurs first for  $\frac{L}{A} \sim 0(Re^{-\frac{1}{2}})$  and therefore consider the limit  $Re \rightarrow \infty$  with

$$G = 2Re^{\frac{1}{2}} \frac{L}{A} \quad (1.2)$$

held fixed. The Görtler number  $G$  defined by (1.2) is, apart from a constant factor, the square of the Görtler number used by Görtler. The relative scales for the velocity components of a Görtler vortex follow from the discussion given by Davey (1962) for narrow gap Taylor vortices so we therefore perturb (1.1) by writing

$$\underline{u} = \underline{u}_B + U_0[U(x, y), Re^{-\frac{1}{2}}V(x, y), Re^{-\frac{1}{2}}W(x, y)]\expiaz. \quad (1.3)$$

Here  $a$  is the vortex wavenumber and the spanwise variable  $z$  has been scaled on the boundary layer thickness. Thus the normal and spanwise velocity components of the vortex are of size  $Re^{-\frac{1}{2}}$  smaller than the downstream component. We have assumed above that the perturbation is steady; such an assumption is valid only for the initial linear stages of the vortex development and is consistent with experimental observations. Recent work by Park and Huerre (1988) on a model problem of some possible relevance to the Görtler problem suggests that the Görtler mechanism is a convective one rather than an absolute instability so that (1.3) is the appropriate form for the vortex in the linear regime. The perturbation is assumed small enough for linearization to be a valid procedure so that

substitution of (1.3) into the Navier-Stokes equations yields:

$$\begin{aligned}
\{\bar{u}\partial_x + \bar{v}\partial_y\}U + U\bar{u}_x + V\bar{u}_y &= \nabla U \\
\{\bar{u}\partial_x + \bar{v}\partial_y\}V + U\bar{v}_x + V\bar{v}_y + G\bar{u}V &= -P_y + \nabla V, \\
\{\bar{u}\partial_x + \bar{v}\partial_y\}W &= -iaP + \nabla W, \\
U_x + V_y + iaW &= 0,
\end{aligned}
\tag{1.4a, b, c}$$

where  $\nabla \equiv \partial_y^2 - a^2$ . Here  $P$  is the pressure perturbation associated with the vortex and a crucial feature of this pressure is that it is absent from the  $x$  momentum equation so that (1.4) is parabolic in  $x$ . However the main feature of (1.4) is that the perturbation is controlled by partial differential equations in  $x$  and  $y$ ; there is no obvious reason why solutions of (1.4) obtained by replacing  $x$ -derivatives acting on perturbation quantities by a constant will have any connection with solutions of the original system. It is the latter assumption which was made by Görtler (1940) and many subsequent authors. It should be remembered that at the time Görtler was performing his calculations it would not have been possible to numerically integrate the partial differential equations. Indeed even the reduced ordinary differential system solved by Görtler was a significant calculation fifty years ago and Görtler obtained approximate solutions by using a Green function technique to reduce the stability equations to an integral equation, furthermore Görtler approximated the basic flow by a piecewise linear profile. Later it was found that Görtler had made an error in his calculations and this was corrected by Hämmerlin (1955).

We refer to any solution of the linear or nonlinear perturbation equations for Görtler vortices which replaces an  $x$ -derivative of a perturbation quantity by a constant as a parallel flow solution; otherwise we refer to it as a nonparallel solution. This terminology

is not ideal because it might be argued that a solution of (1.4) which replaces  $x$ -derivatives acting on  $U, V$  and  $W$  by constants but retains the terms dependent on  $\bar{v}$  does capture some nonparallel properties of the disturbance. However, since such solutions are clearly not valid, and there are many such approximations, it seems pointless to try and attribute them with some validity by describing them in some way which reflects their degree of ‘nonparallelism’.

The disturbance equations (1.4) have been known for some time; perhaps the first derivation of these equations is due to Smith (1955) though the equations are to be found also in Gregory, Stuart and Walker (1955). It appears that Floryan and Saric (1979) were the first to state the equations in the form (1.4) without the retention of formally smaller terms. Even though the correct form for the perturbation equations has been known for some time, it is only in the last decade that solutions of (1.4) which correctly take care of the streamwise structure have been found. In the next section we will describe the results obtained by Görtler (1940) and subsequent authors who solved the disturbance equations without taking care of the  $x$  dependence of the disturbance velocity field in a self-consistent manner.

In Section 3 we derive nonparallel solutions of the disturbance equations for  $a \gg 1$  and discuss numerical solutions of the full system. The numerical solutions of the full system do not lead to a unique neutral curve because of the influence of initial conditions. Thus it is somewhat ironic that the forty year long search for the neutral curve for Görtler vortices was necessarily doomed to fail because the concept of a unique neutral curve is not tenable for the Görtler problem. However we note that the different neutral curves

predicted by the parallel flow curves disagree because of the inconsistent retention of some nonparallel effects and higher order curvature effects whereas the nonuniqueness associated with the full system is associated with its parabolic nature. In that section we also present some results of what is apparently the first investigation of the receptivity problem for Görtler vortices. Here we find that in, some sense, a unique neutral curve for Görtler vortices can be found for a quite general class of incoming disturbances.

In Section 4 we shall discuss nonlinear aspects of the Görtler problem. It is at this stage that a major difference between the Görtler instability and other instabilities develop. In particular we find that, for small vortex wavelengths, perhaps uniquely in fluid dynamics, the onset of nonlinear effects close to the position of neutral stability is not governed by a Stuart-Watson amplitude expansion. It turns out that when Görtler vortices become nonlinear there is a mean field interaction between the fundamental mode and the mean flow correction; there is not a cascade of energy into the higher harmonics. Thus the onset of nonlinear effects in the Görtler problem leads to a pair of coupled nonlinear partial differential equations rather than an ordinary differential equation for the disturbance amplitude. These evolution equations are valid sufficiently close to the position where instability first occurs; however, they point to the existence of a remarkably simple fully nonlinear state further downstream. This fully nonlinear state has close connections with the so-called marginal theory of turbulence proposed in a different context by Malkus (1956). In particular the mean (independent of  $z$ ) part of the flow in the nonlinear state turns out to satisfy an equation which enables the fundamental and all higher harmonics of the vortex to remain neutrally stable. Thus, where vortices exist, the mean state no

longer satisfies the boundary layer equations. It turns out that the vortices decay to zero in shear layers away from the centre of vortex activity and beyond these shear layers the mean state satisfies the boundary layer equations. Even further downstream it is possible to describe the three-dimensional time dependent breakdown of these vortices as a wavy vortex mode becomes unstable in either of the shear layers. It is somewhat surprising that the Görtler problem can be described asymptotically in a strongly nonlinear regime way beyond what is possible for apparently much simpler instabilities such as Taylor vortices or Bénard convection.

Finally in Section 5 we shall draw some conclusions and briefly discuss some recent results on vortex-wave interaction theory.

## 2. Parallel flow theories of Görtler vortex growth.

In 1923 G.I. Taylor had shown conclusively that centrifugal instabilities between rotating concentric cylinders could be accurately described in the linear regime by a stability theory which took viscous effects into account. On the basis of inviscid theory it was known from Rayleigh (1916) that flows with curved streamlines are locally inviscidly unstable if the circulation decreases in a direction away from the local centre of curvature. The latter result suggests that a Blasius boundary layer is centrifugally unstable if the wall is concave and stable otherwise. Other boundary layer flows such as the wall jet can be unstable on both convex and concave walls. The instability, when it occurs, takes the form of counter-rotating streamwise vortices known as Taylor or Görtler vortices depending on whether or not the basic state is fully developed.



In 1940 Görtler formulated the linear stability problem for a two-dimensional boundary layer on a curved wall. He ignored the spatial development of the boundary layer and the normal velocity component associated with that flow. If  $U_0$  is the free-stream speed,  $\nu$  the viscosity of the fluid,  $\delta$  the boundary layer thickness and  $A$  the radius of curvature, then following Görtler we define a Görtler number by

$$G_\theta = \frac{U_0 \delta}{\nu} \left( \frac{\delta}{A} \right)^{\frac{1}{2}} \quad (2.1)$$

which, apart from a constant, is the square root of the Görtler number defined by (1.2).

The approximations made by Görtler therefore led him to consider the eigenvalue problem

$$\begin{aligned} \left\{ \frac{d^2}{dy^2} - a_\theta^2 - \sigma \right\} U &= G_\theta V \frac{d\bar{u}}{dy} \quad , \\ \left\{ \frac{d^2}{dy^2} - a_\theta^2 - \sigma \right\} \left\{ \frac{d^2}{dy^2} - a_\theta^2 \right\} V &= -2a_\theta^2 G_\theta \bar{u} U \end{aligned}$$

subject to  $U = V = V_y = 0, y = 0$ . Here  $a_\theta$  and  $\sigma$  are the spanwise wavenumbers and temporal growth rate of the vortex whilst  $\bar{u}(y)$  is the local approximation to the streamwise boundary layer velocity. Görtler obtained a solution by using Greens functions to transform the eigenvalue problem into an integral equation which he then solved numerically. Later Meksyn(1950) used a WKB approximation procedure to solve Görtlers equations and the neutral curves found by Meksyn and Görtler were similar to those for Taylor vortices and Benard convection. Their results suggested that, on the assumption that boundary layer growth is not important, instability occurs first at a finite value of the vortex wavenumber at some critical Görtler number. Subsequently it was found that there was an error in Görtler's calculations which was corrected by Hämmerlin (1955) who resolved Görtler's equations. The neutral curves,  $\sigma = 0$ , obtained by Görtler (1940) and Hämmerlin (1955)

are shown in Figure 2.1. We see that the correct solution of the simplified equations predicts that instability occurs first at zero wavenumber, in which case the vortices extend beyond the edge of the boundary layer. That result contradicts the assumption that the vortices are confined to the boundary layer so it was argued by various authors that finite curvature or nonparallel effects might be used to remedy this deficiency in the theory. Thus there followed in the next twenty years a series of papers motivated primarily to correct the zero wavenumber degeneracy of Görtler's theory.

Perhaps the most significant of these calculations was the work of Smith (1955) who devised a modified form for the eigenvalue problem and chose to look at spatial instability rather than temporal instability. The equations derived by Smith took some account of streamline curvature and retained the terms associated with the nonzero normal velocity component in the boundary layer. The equations solved by Smith produced a critical Görtler number at a finite wavenumber. It was significant that Smith clearly recognized that, if the non-neutral theory was to have any relevance for transition prediction, a spatial stability calculation was required.

Meanwhile Hämmerlin (1956) attempted to remedy the deficiency at low wavenumbers by retaining some formally negligible curvature effects. A related calculation was then carried out by Witting (1958) who confirmed Hämmerlin's result that small curvature effects can shift the critical wavenumber to a finite value. The neutral curves obtained by these different approaches were consistent only at high wavenumbers; at low wavenumbers the curves were in marked disagreement. Moreover, since no formal asymptotic justification

could be made for the somewhat arbitrary retention of the apparently small higher order effects, it was not clear which, if any, of these calculations was correct.

The above calculations can be classified as parallel flow calculations because they did not account for every term in the disturbance equations arising from the non-parallel nature of the basic state. Interestingly enough it is worth pointing out that a Stokes layer on a curved surface is susceptible to Görtler vortices and Seminara and Hall (1975) showed that for this parallel boundary layer there is a well defined neutral curve with a left hand branch asymptoting to, rather than crossing, the zero wavenumber axis. That result suggests that the essential difficulty present in the Görtler problem at small wavenumber is a direct consequence of boundary layer growth. We shall see in the next section that if boundary layer growth is taken care of in a self-consistent way the small wavenumber degeneracy is resolved but its resolution causes philosophical problems for the transition prediction fraternity.

A review of many of the early parallel flow theories was given by Herbert (1976). Other aspects of the Görtler problem were investigated by, for example, Kahawita and Meroney (1977), (effect of wall heating), Tobak (1964), (effect of curvature distribution).

More recently Floryan and Saric (1979) reformulated the Görtler problems using streamline co-ordinates in order to overcome the problems associated with the zero wavenumber limit. Even though Floryan and Saric and other previous workers recognized that partial derivatives of the disturbance in the streamwise direction are formally comparable to the terms retained by Görtler, the solution procedure used by them replaced  $\frac{\partial}{\partial x}$  by a constant. Despite Floryan and Saric's claim that this can be justified by the method of

multiple scales, this approximation is not valid because the scales on which the vortex and mean flow develop in the streamwise direction are identical. Thus the solution procedure of Floryan and Saric was incorrect for exactly the same reasons as were previous parallel flow calculations.

Thus in summary we conclude that the parallel flow approaches to the Görtler problem gave inconsistent results at order unity wavenumbers and physically unacceptable results at small wavenumbers. In Figure 2.1 we show a selection of the neutral curves found by different authors. We note that the results appear to be consistent at high wavenumbers; we shall see in the next section that it is an understanding of this regime which enables a self-consistent asymptotic description of Görtler vortices to be carried out.

### 3. Nonparallel linear theory and the receptivity problem.

We shall now discuss how (1.4) can be solved in a manner which takes account of nonparallel effects in a justifiable manner. At this stage we shall make the assumption that the unperturbed state is a Blasius boundary layer. The ideas we discuss are easily applied with some minor modifications to more general boundary layers but for definiteness here we restrict our attention to the zero pressure gradient case. For a discussion of the derivation and solution of the Görtler vortex equations for interactive boundary layers see Hall and Bennett (1986) where, as an illustrative example, the instability of internal boundary layers is discussed.

Next we suppose that we are interested in the spatial evolution of a constant wavelength longitudinal vortex introduced into the flow at, say,  $x = \bar{x}$ . Experimentally it is well known that a vortex conserves its physical wavelength as it develops downstream

so that the non-dimensional wavelength  $2\pi a_x^{-\frac{1}{2}}$  based on the local boundary layer thickness decreases like  $x^{-\frac{1}{2}}$ . The effective local Görtler number,  $G_x$ , for the flow grows like  $\chi(x)x^{\frac{3}{2}}$ , so that in a Blasius boundary layer a constant wavelength vortex develops such that  $G_x \sim a_x^{\frac{3}{2}}\chi(a_x^2)$ . In convective and centrifugal instability theories it is known that the right hand branch of the neutral curve, if one exists, has  $G \sim a^4$ . It follows that if  $\chi(x) \ll x^{\frac{1}{2}}$  for large  $x$  then the vortex must ultimately enter a stable regime. In particular it follows that a Blasius boundary layer over a wall of constant curvature a vortex can and must be unstable for a finite range of values of  $x$ . Moreover, we note that in any growing boundary layer the local wavenumber for a fixed wavelength vortex must grow like the boundary layer thickness. Therefore any initial longitudinal vortex ultimately enters a regime where its effective nondimensional wavenumber is large. We now show how we can exploit this largeness of the local vortex wavenumber to develop a formal asymptotic expansion of (1.4) valid for  $a \gg 1$ .

As one might expect, it is found that at large wavenumbers the vortex feels only the local boundary layer structure and chooses to locate itself where it maximises its downstream growth. Hall (1982a) investigated the solution of (1.4) in the limit  $a \rightarrow \infty$  and expanded  $G$  in the form

$$G = g_0 a^4 + g_2 a^3 + g_2 a^{\frac{7}{2}} + \dots \quad (3.1)$$

where  $\{g_i\}$  are to be found in terms of  $x$  if the growth rate and  $a$  are specified. Alternatively, if we are given  $\{g_i\}$ , then the corresponding growth rate is to be found. A WKB analysis of (1.4) shows that the most unstable longitudinal vortex structures are those which have

a second order turning point behaviour in their vertical structure. This means that the vortices are confined in a layer of depth  $a^{-\frac{1}{2}}$  centred on some location  $y = y^*(x)$ .

In the neighbourhood of  $y^*$  the basic flow and disturbance are expanded in powers of  $a^{-\frac{1}{2}}$  so that for example  $U$  in (1.3) becomes

$$U = \exp\{a^2 \int^x [\beta_0(x) + a^{-\frac{1}{2}}\beta_1(x) + \dots]dx\} \sum_{j=0}^{\infty} a^{-\frac{j}{2}} U_j(x, \xi), \quad (3.2)$$

where

$$\xi = a^{\frac{1}{2}}(y - y^*). \quad (3.3)$$

In (3.2) the functions  $\{\beta_i\}$  determine the spatial growth of the vortex and, after a little work, we find that for example  $\beta_0$  satisfies

$$(\bar{u}^* \beta_0 + 1)^2 = g_0 \bar{u}^* \bar{u}_y^* \chi, \quad (3.4)$$

$$[(\bar{u}^* \beta_0 + 1)^2]_y = g_0 [\bar{u} \bar{u}_y]^*_y \chi, \quad (3.5)$$

and if  $g_0$  is given (3.4), (3.5) can be thought of as equations to determine  $\beta_0$  and  $y^*$ . In that sense (3.5) can be interpreted as the condition which enables the vortex to maximise its spatial growth.

The neutral stability point in the boundary layer can be defined in terms of the zero growth of a particular flow quantity measured at a particular location in the boundary layer. Any such flow quantity has its neutral stability point approximated at zeroth order by  $\beta = 0$  so that (3.5) shows that, where a vortex is neutrally stable, it locates itself at the position where it effectively ‘most violates’ Rayleighs criterion. The expansion procedure

outlined above can be continued to higher order taking care of nonparallel effects in a systematic manner. The vertical structure of the disturbance turns out to be described by parabolic cylinder functions and the only possible breakdown of (3.2) occurs at a point where  $\chi(x)$  changes sign. At such a point  $\beta_0$  is a double eigenvalue and merges into a continuous spectrum, the difficulties associated with connecting (3.2) to an appropriate structure for  $\chi < 0$  have not been resolved, but see Jallade (1989) for some discussion of that problem.

In the neutral case Hall (1982a) showed that for a Blasius boundary layer (3.1) becomes

$$G\chi = 5.91x^{\frac{1}{2}}a^4 + .96a^3 + g_2a^{\frac{7}{2}} + \cdots, \quad (3.6)$$

where  $g_2$  depends on what flow property is used to monitor the growth of the disturbance. This means that the first two terms in (3.6) are given correctly by the various parallel flow theory approximations to (1.4). Thus in the high wavenumber limit the various parallel flow theories become valid but, since this is their only range of validity and the asymptotic approach is at least as accurate and requires no computing whatsoever, this merely demonstrates the futility of solving the parallel flow equations.

For compressible boundary layers the approach of Hall (1982a) can again be used in the high wavenumber limit and Hall and Malik (1989) have presented results similar to those of the incompressible theory. Surprisingly it has been recently found that in the hypersonic limit a Sutherland or Chapman Law fluid has a simplified Görtler structure. Thus Hall and Fu (1989a,b) have shown that in the most dangerous wavenumber regime nonparallel effects are unimportant at zeroth order and that the vortices are trapped in the transition layer where the basic flow temperature field rapidly adjusts to its free stream

value. The results of Hall and Fu were subsequently confirmed by Spall and Malik (1989) who integrated numerically the full partial differential stability equations at finite but large Mach numbers. Earlier work on the compressible Görtler vortex stability problem by Kobayashi and Kohama (1977), El Hady and Verma (1981) had used a parallel flow approximation at finite Mach numbers. Hence these calculations only have any validity in the high wavenumber regime where they are consistent with the Hall and Malik theory.

Now let us turn to the  $0(1)$  wavenumber regime and note that, if  $W$  and  $P$  are eliminated from (1.4) we obtain the following coupled partial differential system for  $U$  and  $V$ :

$$\begin{aligned} U_{yy} - \bar{u}U_x &= a^2U + \bar{u}_xU + \bar{v}U_y + V\bar{u}_y, \\ V\{\bar{u}_{xy} + a^4 + a^2\bar{v}_y\} + \bar{v}_xU_{yy} + \{\bar{u}_{xx} + a^2\bar{v}_x + \chi a^2G\bar{u}\}U \\ &+ \{\bar{u}_{yy} - \bar{u}\partial_y^2 + a^2\bar{u}\}V_x + 2\{\bar{u}_{xy} + \bar{u}_x\partial_y\}U_x \\ &+ V_{yyy} - \bar{v}V_{yy} - \{\bar{v}_y + 2a^2\}V_{yy} + \{\bar{u}_{xy} + a^2\bar{v}\}V_y = 0, \end{aligned}$$

which must be solved subject to the conditions

$$U, V, \frac{\partial V}{\partial y} = 0, \quad y = 0, \infty, \quad (3.8)$$

and, if the vortex is induced by an initial perturbation at  $x = \bar{x}$

$$U = \bar{U}(y), \quad V = \bar{V}(y), \quad x = \bar{x}. \quad (3.9)$$

The functions  $\bar{U}, \bar{V}$  and the position  $\bar{x}$  where the disturbance is introduced are all at our disposal constrained only by the conditions  $\bar{U}''(0) = 0, \bar{U}'''(0) = a^2\bar{U}'(0), \bar{V}''''(0) = 2a^2\bar{V}''(0)$ , which ensure that  $U$  and  $V$  do not have a singularity at  $(x, y) = (\bar{x}, 0)$ . In order



to monitor the growth or decay of the vortices Hall (1983) defined the local growth rate  $\sigma(x)$  by

$$\sigma = \frac{1}{2} \frac{d}{dx} \ln \left\{ \int_0^\infty (U^2 + V^2 + W^2) dy \right\}. \quad (3.10)$$

The position of neutral stability was defined by  $\sigma = 0$  and the local wavenumber calculated at that location. If the initial wavenumber of the imposed disturbance is varied a neutral curve appropriate to a particular initial disturbance can be found. Figure 3.1 shows the dependence of  $\sigma(x)$  on  $\bar{x}$  for the case  $a = .069$ ,  $G = .025$  and  $\bar{U}, \bar{V}$  given by

$$\bar{U} = y^6 e^{-y^2/2\bar{x}}, \quad \bar{V} = 0. \quad (3.11)$$

We observe that the position of neutral stability is indeed a function of  $\bar{x}$  whilst for large  $x$  the growth rates merge because the effective wavenumber is large and the asymptotic theory of Hall (1982a) applies. In Figure 3.2 we show the neutral curves corresponding to different initial conditions (3.11) imposed at  $\bar{x} = 50$ . Also shown in that Figure are some experimental results due to Tani (1962) and Winoto and Crane (1980) together with some parallel flow stability calculations. We see that the nonparallel neutral curves are closer to the experimental results. Figure 3.2 also shows the two-term large wavenumber approximation to the neutral curve, it can be seen that the different approaches merge in that limit. Thus in the only situation where parallel flow theories are valid the asymptotic approach is at least as accurate and trivial to use. We conclude that the concept of a unique neutral curve or growth rate is not tenable for the Görtler problem since the local behaviour of the vortex depends on its upstream form. Whilst such a result is exactly what a theorist would expect, given the parabolic nature of the stability equations, it causes a problem for

design engineers who need a unique growth rate at each  $x$  to feed into their version of the  $e^n$  method for transition prediction. The response to this difficulty has been to criticize the form of the initial conditions chosen by Hall (1983) since the initial velocity field does not look like a parallel flow theory Görtler vortex velocity field, see for example Kalburgi et al (1987,1988), Spall and Malik(1989). It was argued by these authors that the ‘correct’ choice of the initial condition is made by solving the parallel flow stability equations. It has never been clear to the author on what grounds this can be justified; however, it does provide a use for the solutions of the parallel flow eigenfunctions. The criticisms about the choice of initial conditions are unfounded; the point is, of course, that once it is found that Görtler vortices are governed by parabolic partial differential equations, then the downstream behaviour of a vortex is fixed by the upstream structure. Moreover, the appropriate initial conditions are not fixed by the stability equations or a reduced form of these equations. Interestingly it has been found by Swearingen and Blackwelder (1987) that the disturbance velocity field calculated from the partial differential equations is very close to that measured experimentally in the unstable regime.

Suppose next that rather than impose an initial disturbance at a finite value of  $x$  we allow the vortex to be generated by a free stream longitudinal vortex structure impinging on the leading edge of the curved wall. Thus we now address the receptivity problem for Görtler vortices and assume that at the leading edge the  $x$  velocity component of the flow is given by

$$U = 1 + \Delta e^{i\alpha z} U^*(y), \quad (3.12)$$

where we have assumed a dependence of the impinging vortex structure on the boundary layer lengthscale though later we shall see that the ‘most dangerous’ vortex has  $U^*$  independent of  $y$ . We shall now solve the vortex equations in the region  $x \ll 1$  such that (3.12) is satisfied; the initial forms for the  $y$  and  $z$  velocity components are then implied by that calculation.

We assume that the initial disturbance is bounded at infinity so that it is sufficiently general for us to consider the case  $U^*(y) = \cos\{by + \phi\}$  where  $b$  and  $\phi$  are constants so that the disturbance at the leading edge is periodic in the  $y$  and  $z$  directions. In fact this would be the appropriate form for an initial disturbance induced by a grid upstream of the test section in an experiment. At the leading edge of the plate the wavelengths in the spanwise and normal directions are large compared with  $x^{\frac{1}{2}}$ , the scale of the boundary layers there. Thus, as one would expect, it is necessary for us to discuss two regions there, namely the boundary layer  $y \sim x^{\frac{1}{2}}$ , and an outer layer with  $y \sim 0(1)$ . We shall see that in the boundary layer  $y \sim x^{\frac{1}{2}}$  with  $x \ll 1$  the flow responds in a quasiparallel manner to the modulated free stream.

Suppose that we allow  $y/\sqrt{x} \rightarrow \infty$  in the disturbance equations (3.7a,b); after some manipulation we obtain

$$\begin{aligned} \{\partial_y^2 - a^2 - \partial_x - \frac{\beta}{\sqrt{2x}}\partial_y\}U &= 0, \\ \{\partial_y^2 - a^2 - \partial_x - \frac{\beta}{\sqrt{2x}}\partial_y\}\{\partial_y^2 - a^2\}V &= a^2\chi GU + \frac{\beta}{(2x)^{3/2}}U_{yy} + \frac{a^2\beta}{(2x)^{3/2}}U. \end{aligned} \quad (3.13a, b)$$

Here  $\beta$  is the Blasius constant defined by

$$\beta = \lim_{\eta \rightarrow \infty} \{\eta f' - f\}.$$

Since the incoming disturbance is periodic in the  $y$  direction with period  $2\pi/b$  we seek a solution of (3.13a) which maintains that structure. The particular solution of (3.13a) which in the limit  $x \rightarrow 0$  is consistent with (3.11), (3.12) is

$$U = U_p = e^{-\{b^2+a^2\}x} \cos\{by + \phi - b\beta\sqrt{2x}\} \quad (3.14)$$

and the periodic solution of (3.13b) is then

$$V_p = \frac{e^{-\{b^2+a^2\}x}}{b^2+a^2} \cos\{by + \phi - b\beta\sqrt{2x}\} \left\{ \frac{(b^2-a^2)}{\sqrt{2x}} - a^2 G \int^x \chi dx \right\}. \quad (3.15)$$

In fact the  $V$  equation has an eigensolution  $V = Q(x)e^{-ay}$  for arbitrary  $Q(x)$  and we shall see that matching with the boundary layer solution cannot be achieved without this eigensolution. Thus the boundary layer causes the periodic structure of  $V$  to occur only for  $ay \gg 1$  and the appropriate solution of (3.13b) is therefore

$$V = V_p + Q(x)e^{-ay}. \quad (3.16)$$

If we are in the regime where  $x = 0(1)$  then  $ay \gg 1$  at the edge of the boundary layer and so the  $(U, V)$  disturbance equations must be solved subject to

$$(U, V) \rightarrow e^{-\{b^2+a^2\}x} \cos\{by + \phi - b\beta\sqrt{2x}\} \frac{(b^2+a^2, [\frac{(b^2-a^2)\beta}{\sqrt{2x}} - a^2 G \int^x \chi dx])}{(b^2+a^2)}. \quad (3.17)$$

However near the leading edge (3.14), (3.16) apply for  $y = 0(1)$  with  $x \ll 1$  and  $Q(x)$  must be found by matching with the boundary layer solution. Here  $U$  and  $V$  are most easily obtained in the form of expansions in powers of  $x^{\frac{1}{2}}$  from the primitive equations

(1.4) expressed in terms of  $x$  and  $\eta = \frac{y}{\sqrt{2x}}$ . In fact  $U$  and  $V$  are obtained by perturbing the Blasius solution by letting the free stream speed very slightly from unity. We obtain

$$\begin{aligned} U &= \cos \phi \{f' + \eta/2f''\} + \dots, \\ V &= \frac{\cos \phi}{\sqrt{2x}} \left\{ \frac{1}{2}(\eta f' - f) + \eta^2/2f'' \right\} + \dots. \end{aligned} \quad (3.18)$$

When  $\eta \rightarrow \infty$  in (3.18) we obtain

$$\begin{aligned} U &\rightarrow \cos \phi + \dots, \\ V &\rightarrow \frac{\cos \phi}{2\sqrt{2x}} \beta + \dots. \end{aligned} \quad (3.19)$$

Hence if (3.14), (3.16) are to match with the boundary layer solution for  $\eta \gg 1$  we must choose

$$\begin{aligned} Q &+ \frac{\cos[\phi - b\beta\sqrt{2x}]}{b^2 + a^2} \left\{ \frac{(b^2 - a^2)}{\sqrt{2x}} - a^2 G \int^x \chi dx \right\} \\ &= \frac{\beta \cos \phi}{2\sqrt{2x}} + \dots. \end{aligned}$$

This equations for  $Q$  is correct to order  $x^{-\frac{1}{2}}$ . In principle we can continue the above procedure to any order in  $x^{\frac{1}{2}}$  and obtain the higher order terms in the expansion of  $Q$ , for our purposes here it is not necessary to pursue that calculation further.

The small  $x$  solutions for  $y \sim x^{\frac{1}{2}}, y \sim 0(1)$  can then be used to form a composite expansion to give asymptotic forms for  $U$  and  $V$  to begin the numerical solution of (1.4) from some small but finite value of  $x = x_2$ . We restrict our discussion to the case when  $\phi = 0$  which corresponds to the most physically relevant case when  $U^*(0) \neq 0$  (Note that since  $U^* = \cos by \cos \phi - \sin by \sin \phi$  and the problem for  $U, V$  is a linear one the only two distinct cases are  $\phi = 0, \phi = \pi/2$ ).

For the receptivity problem formulated above the disturbance energy is infinite since it is not confined to the boundary layer, therefore we cannot monitor the growth of a

disturbance using the approach of Hall (1983). Since the normal and spanwise velocity components are  $O(Re^{-\frac{1}{2}})$  smaller than the streamwise component it is appropriate to define a local growth rate in terms of  $U$  alone. Thus we use  $\sigma_*$  defined by

$$\sigma_* = \frac{\partial}{\partial x} \{\ln U(x, 0)\},$$

to monitor the growth of the disturbance. We note that if  $\sigma_*$  was used in the calculations of Hall (1983) the results would be virtually identical.

Now let us discuss some results for the Görtler receptivity problem formulated above. The disturbance equations were marched downstream from  $x = x_2$  using the code discussed in Hall(1983). After some experimentation we found that a suitable step length in the streamwise direction was 0.00001 if a normal step length of .0333 was used. All the calculations reported on here correspond to these step lengths and 1000 points were used in the vertical direction. The surprisingly small x-step length was necessary because of the singular behaviour of  $V$  for small  $x$ .

In Figure 3.3 we show  $\sigma_*$  for the case  $\chi = 1., b = 0., G = 70.$  and several different values of the vortex wavenumber  $a$ . For small values of  $x$  the development of the vortex is independent of it's wavenumber and the growth rates are indistinguishable. As the vortex develops downstream the growth rates diverge and become positive at different downstream locations. If the local Görtler number and wavenumber are calculated at the different locations where the growth rate vanishes a neutral curve in the  $(a_x - G_x)$  plane can be calculated. Figure 3.4 shows the result of several such calculations for different values of  $b$ . A crucial result illustrated by this calculation is that instability occurs first for the case  $b = 0.$ , so we conclude that the most dangerous incoming disturbance for the receptivity

problem has  $U^* \sim \cos \alpha z$  at the leading edge of the wall. Our only explanation of why this should be the case is to point out that, if the disturbance was evolving in anything like a quasi-parallel manner, incoming disturbances with the higher values of  $b$  would stimulate the more stable higher Görtler modes.

In Figure 3.5 we show two neutral curves corresponding to  $b = 0$  but with different wall curvatures. This Figure demonstrates that the most dangerous mode is weakly dependent on the wall curvature distribution, this result has implications for the question of how the curvature should be distributed on an aerofoil in order to inhibit Görtler vortex growth.

In Figure 3.6 we have compared our results with the experimental observations of Tani(1962), Bippes and Görtler(1972), Winoto and Crane(1980), and Swearingen and Blackwelder (1987). The curves (a), (c) correspond to a typical neutral curve from Hall(1983) and Floryan and Saric(1979) respectively. The curve (b) comes from the receptivity calculation with  $b = 0, \chi = 1$ ; we stress that this is the most dangerous mode predicted by the receptivity calculation. Apart from the one experimental point below this curve we see that the receptivity calculation is the most consistent with the experiments.

The development of the disturbance velocity components as the vortex develops downstream is shown in Figures 3.7a,b. We note that the edge velocity for the streamwise velocity component decreases monotonically with  $x$  whilst the normal velocity component at the edge of the boundary layer initially decreases but then increases with  $x$ . The initial decrease occurs because  $V \sim x^{-\frac{1}{2}}$  for small enough  $x$  whilst for larger (but not too large)  $x$  the term proportional to  $G$  causes  $V$  to grow. In fact at even larger  $x$  the edge velocity begins to decrease with  $x$  because of the exponential factor in (3.17). It is interesting to

note that whilst the initial streamwise velocity component of the disturbance looks similar to a parallel flow eigenfunction the normal velocity component does not. Indeed in the initial stages of its development this component has sign opposite to that in the unstable regime.

#### 4. The nonlinear stages of Görtler vortex growth

It is in the nonlinear regime that significant differences between Görtler vortices and other hydrodynamic instabilities occur. We recall that for most fluid flows the onset of nonlinearity can be described by the Stuart-Watson method. This method shows that in the nonlinear state energy cascades down from the fundamental into the higher harmonics and the mean flow. At small disturbance amplitudes  $A$  it is found that  $A$  satisfies an equation of the form

$$\frac{d}{dt}|A|^2 = \mu|A|^2 \pm |A|^4, \quad (4.1)$$

where  $\mu$  is a prescribed real constant,  $t$  a slow time variable and the  $+$  or  $-$  sign is to be taken dependent on whether nonlinear effects are stabilizing or destabilizing. The constant  $\mu$  is positive or negative dependent on whether the basic state is linearly unstable or stable. In the Taylor vortex problem nonlinear effects are stabilizing so that (4.1) has the stable finite amplitude equilibrium solution

$$|A|^2 = \mu. \quad (4.2)$$

Thus in this situation the small but finite amplitude disturbance is determined by an interaction involving the fundamental, mean flow correction and the first harmonic. The



secondary instability of the Taylor vortex state (4.2) to wavy vortex instabilities was discussed in the paper by Davey, Di Prima and Stuart (1968). The wavy vortex mode arises from the linear instability of a steady axisymmetric vortex flow to a time-periodic non-axisymmetric mode disturbance and can be described by a generalization of (4.1) to two coupled amplitude equations. Experimentally it has been observed by, for example Aihara and Kohama (1981), that Görtler vortices undergo a similar breakdown at finite amplitude. The similarity between the Görtler and Taylor problems suggests that the procedure used so successfully by Davey, Di Prima and Stuart (1968) should, subject to some nonparallel modifications, be able to describe the breakdown of Görtler vortices.

The first nonlinear calculation of Görtler vortices known to the author is that due to Aihara (1976). That calculation ignored nonparallel effects and attempted to simulate nonlinear effects by an approximate averaging technique. The procedure has some similarity with the Stuart-Watson method but no formal amplitude equation was derived. Aihara claims that his analysis gives predictions consistent with his experimental results.

We have seen already that in the linear regime nonparallel effects cannot be ignored at  $O(1)$  vortex wavenumbers. This must also be the case when nonlinear effects are taken into account and Hall (1988) integrated the nonlinear version of (1.4) for a variety of different disturbance amplitudes. The method used was based on that of Hall (1983) but with an iteration procedure to take care of the nonlinear terms. At sufficiently small disturbance amplitudes the calculation reproduced the linear results of Hall (1983) whilst at larger initial amplitudes finite amplitude states were calculated. In Figure 4.1 we show some results from Hall (1988) for a wall with curvature distribution  $\chi \sim x$  with  $G =$

.0288,  $a = .1$ . The energy in the fundamental and the mean flow correction due to the vortex is shown for different values of the vortex amplitude  $\Delta$ . We note that the calculations were for an initial disturbance such that  $\Delta$  represents the maximum value of the streamwise disturbance velocity component divided by the free stream speed. In any given calculation it was found that, sufficiently far downstream, the only energy present in the disturbance velocity field was associated with the fundamental or the mean flow correction. The total downstream velocity component at different spanwise locations is shown in Figure 4.2 for  $\Delta = .1$ . Here we see that, at the spanwise location where upwelling occurs, highly inflectional velocity profiles are set up as the vortex develops in  $x$ . These profiles are locally unstable to inviscid Rayleigh waves (see Horseman (1990)) so that in some experimental situations we expect that the onset of time-dependence in the Görtler problem will occur as an inviscid secondary instability of a finite amplitude stationary vortex.

The nonlinear calculations of Hall (1988) suggest that any initial spanwise vortex distribution ultimately develops into a finite amplitude state in which the only energy interchange is between the fundamental and the mean flow. We now describe strongly nonlinear stability calculations, due to Hall and Lakin (1988), which explains the latter type of interaction. Earlier Hall (1982b) had investigated the weakly nonlinear regime corresponding to small wavelength Görtler vortices. It was found that, within an  $a^{-1}$  neighbourhood of the linear neutral location the initial development of the vortex is governed by the system

$$\begin{aligned} \left\{ \frac{\partial^2}{\partial \xi^2} - \frac{1}{4} \xi^2 - k \frac{\partial}{\partial \bar{x}} \pm \bar{x} \right\} U &= V \frac{\partial \bar{u}}{\partial \xi}, \\ \frac{\partial^2 \bar{u}}{\partial \xi^2} - \frac{\partial \bar{u}}{\partial \bar{x}} &= \frac{\partial}{\partial \xi} (V^2). \end{aligned} \tag{4.3a, b}$$

Here  $k$  is a constant,  $a^{-1}V \cos az$  is the normal velocity component of the vortex whilst  $a^{-1}\bar{u}$  is the mean flow correction. The variable  $\bar{x}$  is defined by

$$\bar{x} = a(x - x_N)$$

where  $x_N$  is the neutral location and  $\xi$  is as defined by (3.3). The higher harmonics generated are negligible in the interaction so that the initial nonlinear evolution of a vortex of small wavelength certainly occurs as a mean field interaction. The positive or negative sign in (4.3a) is to be taken dependent on whether the curvature of the wall increases more or less quickly than  $x^{\frac{1}{2}}$ . On a concave wall of constant curvature the negative sign is appropriate and then (4.3) describes the finite amplitude decay of a vortex.

In Figure 4.3 we show some numerical solutions of (4.3) subject to the conditions

$$V \rightarrow 0, \quad \bar{u} \rightarrow 0, \quad |\xi| \rightarrow \infty,$$

so that the effect of the vortex is confined to the  $\xi = 0(1)$  layer. Figure 4.3 suggests that for large  $\bar{x}$  the functions  $\bar{u}, V$  develop an asymptotic structure with  $V$  trapped in a layer of depth  $\bar{x}^{\frac{1}{2}}$  whilst  $\bar{u}$  decays to zero over a lengthscale  $0(\bar{x}^{\frac{3}{2}})$ . In this large  $\bar{x}$  region  $V$  and  $\bar{u}$  are respectively symmetric and axisymmetric about  $\xi = 0$ . Hall(1982b) shows that the vortex activity decays to zero as the solution of a nonlinear Airy equation in layers of depth  $\bar{x}^{\frac{1}{2}}$  situated symmetrically at distance  $0(\bar{x}^{\frac{1}{2}})$  away from  $\xi = 0$ . The mean flow correction in these layers does not develop a structure and is therefore reduced to zero in a thicker layer of depth  $\bar{x}^{\frac{3}{2}}$ .

A result of some significance is that for large  $\bar{x}$  the mean flow correction is of size  $\bar{x}^{\frac{3}{2}}$  and since  $\bar{u}$  was initially scaled with  $\bar{a}^{-\frac{3}{2}}$  it follows that when  $\bar{x} = 0(a)$ , i.e. when

$x = x_n = 0(1)$ , the mean flow correction will be comparable with the original basic state. In nonlinear stability problems governed by the Stuart-Watson method a related stage is encountered but all the harmonics of the fundamental then become equally important so that no asymptotic description of this strongly nonlinear regime is available. In the Görtler problem this is not the case and surprisingly it was shown by Hall and Lakin (1988) that, even in the strongly nonlinear regime, where the basic state is totally restructured by the vortex, an asymptotic description of the interaction is available. The structure found by Hall and Lakin can be written down as the limiting form of the weakly nonlinear solution of Hall (1982b) which produces an  $0(1)$  correction to the linear flow. For convenience we now let  $\epsilon$  denote  $a^{-1}$ . The different regions found in that limit are illustrated in Figure 4.4. In region I a finite amplitude vortex exists which is large enough to generate a transfer of energy into the mean state thereby determining that state. In the thin shear layers IIa,b the vortex activity is reduced to zero again through the solution of a nonlinear Airy equation. In regions IIIa,b there is no vortex activity and the mean flow satisfies the boundary layer equations. However, the solutions of the boundary layer equations in IIIa,b must satisfy certain conditions at  $y_1(x), y_2(x)$ , the unknown positions of the shear layers IIa,b.

In region I the total  $x$  and  $y$  velocity components expand as

$$\begin{aligned}
u &= \bar{u}_0 + \epsilon \bar{u}_1 + \cdots + [\{\epsilon E U_0^1 + \epsilon^2 E U_1^1 + \cdots\}, \\
&\quad + \epsilon^2 E^2 U_0^2 + \cdots] + c.c \\
v &= \bar{v}_0 + \epsilon \bar{v}_1 + \cdots + [\{\epsilon^{-1} E V_0^1 + E V_0^1 + \cdots\}, \\
&\quad + E^2 V_0^2 + \cdots] + c.c
\end{aligned} \tag{4.4a, b}$$

where  $E = \exp(iaz)$ .

The Görtler number expands as

$$G = G_0 a^4 + \dots \quad (4.5)$$

If the above expansions are substituted into the Navier-Stokes equations and the dominant terms are equated we obtain

$$\begin{aligned} \frac{\partial V_0^1}{\partial y} + iW_0^1 &= 0, \\ U_0^1 + V_0^1 \frac{\partial \bar{u}_0}{\partial y} &= 0, \\ V_0^1 + G_0 \chi U_0^1 \bar{u}_0 &= 0, \\ -iP_0^1 &= W_0^1. \end{aligned} \quad (4.6a, b, c, d)$$

If  $U_0^1$  and  $V_0^1$  are known then equations (4.6a,d) determine  $W_0^1$  and  $P_0^1$  respectively but (4.6b,c) do not determine the former functions. In fact (4.6b,c) only have a consistent solution if

$$G \chi \bar{u}_0 \frac{\partial \bar{u}_0}{\partial y} = 1, \quad (4.7)$$

which determines the mean state which can support the imposed vortex structure. Indeed (4.7) can then be interpreted as the equation which determines a basic state which makes the vortex with wavenumber  $a$  and all its harmonics neutrally stable. This behaviour is exactly that postulated by Malkus (1956) who argued that in a turbulent flow the mean state was that which made all modes present neutrally stable. In other words what Hall and Lakin found was that, when nonlinear stability theory is pushed way beyond the weakly nonlinear state, the mean state where vortices exist subtly arranges itself so that small wavelength vortices are neutrally stable. The solution of (4.7) is then given by

$$\bar{u}_0 = \frac{\sqrt{a(x) + 2y}}{\sqrt{G_0 \chi}}, \quad (4.8)$$

where  $a(x)$  is an unknown function of  $x$ .

The  $y$  component of the mean state is then determined by the continuity equation to give

$$\bar{v}_0 = -\frac{a'\sqrt{a+2y}}{2\sqrt{G_0\chi}} + \frac{(a+2y)^{\frac{3}{2}}\chi'}{6\sqrt{G_0\chi}} - b(x). \quad (4.9)$$

Here  $b(x)$  is another unknown function of  $x$ . Thus the mean state in I is determined by insisting that the equation satisfied by the vortex in the core should have a consistent solution.

Meanwhile the  $x$ -momentum equation yields the following equation when the dominant terms independent of the spanwise variable are retained:

$$\bar{u}_0 \frac{\partial \bar{u}_0}{\partial x} + \bar{v}_0 \frac{\partial \bar{u}_0}{\partial y} - \frac{\partial^2 \bar{u}_0}{\partial y^2} = 2 \frac{\partial}{\partial y} \{ \bar{u}_0 |V_0^1|^2 \}. \quad (4.10)$$

At this stage  $\bar{u}_0$  and  $\bar{v}_0$  are already known so that (4.10) determines  $|V_0^1|^2$ . If (4.10) is integrated with respect to  $y$  we obtain

$$B(x) - 2 \frac{\partial \bar{u}_0}{\partial y} |V_0^1|^2 = \frac{b\sqrt{a+2y}}{\sqrt{G_0\chi}} + \frac{1}{12} \frac{(a+2y)^2 \chi'}{G_0\chi^2} + \frac{1}{\sqrt{a+2y}\sqrt{G_0\chi}}, \quad (4.11)$$

where  $B(x)$  is another function of  $x$  to be determined along with  $a(x)$  and  $b(x)$ . The phase of  $V_0^1$  is determined from the spanwise momentum equation but is not needed in the present discussion. Since  $|V_0^1|$  cannot be negative (4.11) can be used to determine  $y_1(x), y_2(x)$  the locations where the vortex activity decays to zero. Thus (4.11) is satisfied at  $y_1, y_2$  with  $V_0^1 = 0$  and if  $B(x)$  is then eliminated we obtain an equation of the form

$$F(a, b, y_1, y_2, \chi, G_0) = 0. \quad (4.13)$$

In fact (4.13) above is not sufficient to determine  $y_1$  and  $y_2$  since  $a$  and  $b$  are also unknown.

An analysis of the shear layers IIa,b show that the vortex activity which is decaying algebraically on entering the layers is reduced to zero exponentially as the solution of the Painlevé equation

$$\frac{d^2\psi}{dy^2} - y\psi = \psi^3, \quad (4.14)$$

$$\psi \sim \sqrt{-y}, \quad y \rightarrow -\infty.$$

The mean flow functions do not acquire any structure in the transition layer scale at leading order so that  $\bar{u}_0$ ,  $\bar{u}_{0y}$  and  $\bar{v}_0$  remain constant in IIa,b.

Finally in IIIa,b there is no vortex motion so that the mean flow  $\bar{u}, \bar{v}$  at leading order satisfies

$$\begin{aligned} \bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} &= \frac{\partial^2 \bar{u}}{\partial y^2}, \\ \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} &= 0 \end{aligned} \quad (4.15)$$

which must be solved subject to

$$\begin{aligned} \bar{u} = \bar{v} = 0, \quad y = 0, \\ \bar{u} \rightarrow 1, \quad y \rightarrow \infty. \end{aligned} \quad (4.16)$$

However (4.15) is valid only in  $(0, y_1), (y_2, \infty)$  so the problem is completed by the conditions

$$\begin{aligned} \bar{u} = \frac{\sqrt{a + 2y_j}}{\sqrt{G_0\chi}}, G_0\bar{u}\bar{u}_y\chi = 1, \bar{v} = \frac{a'\sqrt{a + 2y_j}}{2\sqrt{G_0\chi}} + \frac{(a + 2y_j)^{\frac{3}{2}}\chi'}{6\sqrt{G_0\chi\chi}} - b, \\ y = y_j \quad \text{for } j = 1, 2. \end{aligned}$$

Furthermore the ‘jump’ condition (4.13) must also be satisfied so that the mean flow is determined as the solution of the boundary layer equations subject to conditions at two unknown interfaces  $y_1$  and  $y_2$ . This numerical problem can be reduced to the solution of an ordinary differential system in the special case  $\chi \sim x^{\frac{1}{2}}$  since  $\bar{u}$  and  $\bar{v}$  then depend on  $x$  and the similarly variable  $yx^{-\frac{1}{2}}$ . In general the partial differential system must be solved and this was done by mapping  $(0, x_2)$  into  $(0, 1)$  and  $(y_2, \infty)$  into  $(1, \infty)$  and solving the

boundary layer equations in  $(0, \infty)$  subject to jump conditions at  $J = y/y_j = 1$ . The  $y$ -derivatives were approximated using finite differences and the calculations were started by evaluating the weakly nonlinear form of the solution as  $x \rightarrow x_n$ .

Figure 4.5 shows  $y_1$  and  $y_2$  obtained from the above scheme for the case  $G_0 = \frac{1}{2}, \chi = \sqrt{2x}$ . Also shown in this Figure are the results obtained from the similarity solution. Figure 4.6 shows  $y_1$  and  $y_2$  for the case  $\chi = 2x, G_0 = 4.176$ . In that Figure we have also shown asymptotic solutions for  $y_1$  and  $y_2$  which can be readily obtained in the limits  $x \rightarrow x_n, x \rightarrow \infty$ . In Figure 4.7 the mean velocity component is shown at  $x = 1$  as a function of  $y$  for  $\chi = 2x, G_0 = 4.176$ . These results demonstrate that for large  $x$  the region of vortex activity spreads throughout the boundary layer; more precisely Hall and Lakin (1988) show that if  $\chi \sim x^M$  with  $M > \frac{1}{2}$  then the free boundary problem for  $y_1$  and  $y_2$  yields

$$y_1 \sim x^{2-3M}, \quad y_2 \sim x^M, \quad x \rightarrow \infty.$$

Thus the presence of large amplitude vortices causes the mean state to be altered almost everywhere from its unperturbed form. Indeed the boundary layer now grows like  $x^M$  rather than  $x^{\frac{1}{2}}$  so that it is thickened by the presence of the vortices. We further note that if  $M > \frac{2}{3}$  the lower transition layer approaches the lower wall and then the mean downstream velocity is a simple linear shear flow from  $y = 0$  to  $y_1$ . We note that at some stage the layer IIa becomes of comparable depth to IIIa when  $x \rightarrow \infty$  so that a modified Painlevé problem must be solved there, Blennerhassett and Bassom, private communication.



Thus, perhaps uniquely in hydrodynamic stability theory, we see that small wavelength Görtler vortices can be described asymptotically all the way from the linear regime to a nonlinear state where the mean flow is driven almost everywhere by the vortices. For some flows the Görtler vortex mechanism is operational only for a finite range of values of  $x$ ; for a Blasius boundary layer this is the case if the wall curvature does not increase as quickly as  $x^{\frac{1}{2}}$ . In that situation the free boundary problem specified above will terminate at some  $x = x_T$  where  $y_1$  and  $y_2$  coalesce. Beyond  $x = x_T$  there is no vortex activity and the flow is obtained by solving the boundary layer equations throughout  $(0, \infty)$ . However, even though the vortex activity has ceased, the mean state will not be the same as that which is set up in the absence of upstream vortices.

The main result then of the Hall-Lakin analysis is that where large amplitude vortices exist they drive the mean state which must satisfy (4.7). In fact (4.7) also applies to curved channel flows. A modified form of (4.7), namely

$$G_0 \chi \bar{u}_0 \frac{d\bar{u}_0}{dy} - \left( \frac{d\bar{u}_0}{dy} \right)^2 = 1, \quad (4.17)$$

was shown by Bassom and Hall (1988) to determine the mean state in curved channel flows driven by the interaction of vortices and Tollmien-Schlichting waves. Interestingly, it was found that in curved channel flows there can be no large amplitude vortex state in the absence of curvature since (4.17) with  $G_0 = 0$  has no acceptable solutions. This result suggests that Görtler and Witting's (1958) suggestion that Görtler vortices occur as secondary instabilities of Tollmien-Schlichting waves is not correct. Finally, before going

on to discuss the breakdown of the nonlinear state of Hall and Lakin (1988) we point out that a further generalization of (4.7), namely

$$R_0 \frac{\partial \bar{T}}{\partial y} = -1,$$

where  $R_0$  is a scaled Rayleigh number and  $\bar{T}$  a mean temperature is appropriate to Bénard convection problems where the instability is initially localized in the vertical direction.

Now let us turn to the instability of the vortex states found by Hall and Lakin (1988). Experimentally it has been known since the work of Bippes (1972) that finite amplitude vortices can be unstable to a wavy vortex disturbance propagating in the streamwise direction. Hall and Seddougui (1989) have recently investigated this possibility and show that if this type of secondary instability is present in the strongly nonlinear regime of Hall and Lakin then it must be concentrated in either of the shear layers IIa,b.

The disturbance imposed on the flow in regions IIa,b by Hall and Seddougui is  $\pi/2$  radians out of phase with the primary vortex and is proportional to

$$\exp ia^2 \left\{ \int^x K(x) dx - \Omega t a^2 \right\}.$$

Here  $\Omega$  is the fixed (real) frequency of the wavy mode whilst  $K(x)$  is a complex wavenumber which will evolve in  $x$  as it develops in the streamwise direction. In fact  $K(x)$  is found to expand as

$$K = K_0(x) + a^{-\frac{2}{3}} K_1(x) + \dots$$

and  $K_0$  turns out to be purely real and such that the wave propagates downstream with the mean velocity field in IIa or IIb. The next term  $K_1(x)$  is complex in general but at particular locations and frequencies is real. Hall and Seddougui indentified a number of

such neutral states and showed that in general the upper shear layer will break down first; note that since the wavy mode is trapped in IIa or IIb there is no reason why the latter layers should simultaneously become unstable to the wavy mode. The results of Hall and Seddougui were entirely consistent with the experimental observations of Peerhossaini and Wesfreid (1988) who identified low and high frequency secondary instability breakdowns of the flow below and above the region of vortex activity. The type of flow pattern predicted by the Hall-Seddougui calculation after the secondary instability is shown in Figure 4. 8.

An alternative description of nonlinear Görtler vortices has recently been given by Sabry and Liu (1987). The latter authors made a parallel flow approximation and modelled the spatial growth of a boundary layer by letting it evolve in time. This procedure is often used in Computational Fluid Dynamics when the instability of a spatially varying flow is simulated numerically. The basic state used in such calculations is a solution of the Navier-Stokes equations only if some fictitious body force is applied. Some justification for this approach is often made by appealing to Gaster's (1962) discussion of spatial and temporal growth rates and their relationship. In situations where transition is dominated by Tollmien-Schlichting waves it appears that numerical situations carried out using this approach are remarkably successful in reproducing experimental results, so that, even though no formal justification for the technique can be given, this type of parallel flow temporal simulation captures the essential physics of transitional flat plate boundary layer. However for Tollmien-Schlichting waves it is known that instability occurs at relatively high Reynolds numbers; in this case the parallel flow approximation can be formally justified

and therefore it is not unreasonable to assume that the essential physics of the problem is obtained within the framework of this approximation.

As we have seen in Section 3 and this section, a crucial property of Görtler vortices is that at  $O(1)$  wavenumbers their evolution is completely controlled by nonparallel effects. This suggests that temporal parallel flow simulations of nonlinear Görtler vortices might well be of little relevance to the real problem.

Sabry and Liu compared the results of their calculations to the experiments of Swearingen and Blackwelder (1987). As an initial disturbance they introduced a parallel flow eigenfunction of amplitude appropriate to the experiments. The parallel flow equations were then marched forward in time and related to the spatial case by a convenient choice of the convection velocity. This velocity was chosen in order to optimize the agreement with theory and experiment. Note that in this type of simulation the boundary layer thickness is a function of time so the instantaneous effective wavenumber also varies in time. The Sabry-Liu calculations were begun at a position where the experimentally observed vortices were certainly nonlinear. Figures 4.9 and 4.10 compare the displacement thickness and wall shear at the peak-valley locations as predicted by Sabry and Liu and measured by Swearingen and Blackwelder. Figure 4.9 also shows the wall shear for Blasius flow whilst the Blasius wall shear and the shear for a turbulent boundary layer are shown in Figure 4.10. The agreement between the calculations and the experiments is exceptionally good ; a few words of caution though are perhaps appropriate.

Firstly, it should be pointed out that the Sabry-Liu calculation does not allow for any streamwise dependence of the disturbance , thus necessarily they cannot capture the

secondary instability of the initial vortex state. Almost certainly the turning points in the experimental data occur when the initial vortex state has broken down; it is therefore surprising that this effect is captured by Sabry and Liu. Secondly it is not clear from the Sabry-Liu calculation how the agreement between theory and experiment depends on the two parameters at their disposal, i.e. the convection velocity and the initial shape and location of the disturbance.

Also shown in Figures 4.9, 4.10 are the results obtained using the code of Hall (1988) to simulate the experiments. The calculations were started at a position where the measured vortex was small, we see that the spatial calculations correctly predict the right trends in shear and displacement thickness up to the point where the corresponding quantities measured experimentally develop turning points. We believe that this is to be expected since at that stage the vortices have become unstable to time-dependent perturbations; interestingly the velocity profiles just before this happens are highly inflectional and so inviscid instabilities would possibly cause a secondary instability to occur. We note that the spatial calculation suggests that the wall shear at the peak locations is about to change sign when the calculations were stopped; this was done because at this stage our results began to develop a grid-size dependence which was presumably caused by the local sign change of the downstream velocity component. We further note that the spatial code started from different initial stations gave quantitatively similar results to those reported above. Thus our spatial calculations suggest that The Sabry-Liu calculations should be treated with some caution. A much more detailed comparison between temporal and spatial nonlinear simulations of Görtler vortices has been made by Malik (1989), private

communication. Malik used the spatial code of Hall (1988) and a temporal code derived from Malik, Zang, and Hussaini(1985). Malik was unable to reproduce the results of Sabry and Liu from his temporal simulations; indeed Malik's temporal simulations were qualitatively similar to those from the spatial approach.

## 5 Conclusions

We have seen that the linear evolution of Görtler vortices in growing boundary layers is dominated by nonparallel effects except in the small wavelength limit. In the latter regime a simple asymptotic description of the vortices is available whilst at bigger wavelengths the linear partial differential equations governing the linear growth of vortices must be solved numerically. The early work in this field ignored the nonparallel effects possibly because quasi-parallel stability theory had been so successful in explaining the growth of Tollmien-Schlichting waves in boundary layers. The reason why parallel flow theory captures the essential details of Tollmien-Schlichting wave growth is that this instability occurs at relatively large Reynolds numbers at a wavelength small compared to a typical distance over which the boundary layer itself evolves. Thus the early parallel flow stability calculations of Tollmien-Schlichting waves appear as the first approximation of the procedures devised by Bouthier(1973) and Gaster(1974). Similarly, in the more formal asymptotic description of Tollmien-Schlichting waves by Smith(1979), nonparallel effects do not appear at leading order. For the vortex instability mode of a boundary layer it is only at high wavenumbers that the spatial evolution of the instability occurs on a shorter lengthscale than that over which the mean flow develops.

Another significant difference between the Görtler and Tollmien-Schlichting modes is that they are respectively governed by elliptic and parabolic differential equations. This means that the position of neutral stability of a vortex depends on its upstream behaviour. This latter property was the main result found by Hall(1983); later researchers attempted to avoid the troublesome non-uniqueness implied by that property by arguing that the parallel flow eigensolutions should be used as 'the initial conditions' for the disturbance equations. In this paper we have shown that if, the instability arises from a longitudinal vortex structure impinging on the leading edge of the curved wall, then a most dangerous mode can be isolated and its neutral curve calculated. The curve we have found is certainly much more in line with experimental observations than those found previously. However, in some experiments the vortices could be tripped by disturbances introduced at the wall, and in that case we would expect a different neutral curve to exist. Thus, as a summary of our receptivity results, we can say that if a Görtler experiment is performed in a facility which allows the instability to be triggered by free stream disturbances which include all possible spanwise wavelengths, then the experimental results should correlate with the lowest neutral curve from the receptivity calculations. It should be noted that the neutral curves found in the receptivity calculations depend on the rate of change of the Görtler number so that the neutral curve of the most dangerous mode needs to be calculated for any particular curvature distribution. Also it needs to be pointed out that the receptivity calculation we have carried out can be extended to any two-dimensional boundary layer but the initial form of the instability is a function of the boundary layer.

All the results discussed in this paper are for two-dimensional boundary layers. The only work on Görtler instabilities in three-dimensional boundary layers known to the authors is that due to Hall(1985). That paper investigated the instability of weakly three-dimensional boundary layers on infinite swept walls where the spanwise velocity component was comparable to the normal velocity component in the boundary layer. It was found that, as the spanwise velocity component is increased, small wavelength vortices are stabilized and become time-dependent. The results suggest that, in a three-dimensional boundary layer, the Görtler mechanism might not even be operational. However a word of caution is in order because the order one wavenumber problem is yet to be tackled. In fact there is some limited experimental evidence, Baskaran and Bradshaw(1988), for turbulent three-dimensional boundary layers which suggests that a crossflow can destroy the Görtler mechanism.

Finally, in our discussion of results for linear theory, we point out that some work has been done on boundary layer flows other than the Blasius one. Thus for example the wall-jet has been studied by number of authors. This boundary layer is of practical interest and can be unstable on both concave and convex walls, Drazin and Reid(1979) and Floryan (1986).

The reader is referred to the latter paper for a discussion of parallel flow theory applied to the wall-jet problem and to Wadey(1990) for the nonparallel and nonlinear situations. Here we merely report that nonparallel effects again dominate the linear growth of the vortices for wall jets and that in the nonlinear regime the approach of Hall and Lakin(1988) can be used.



It is in the nonlinear regime where the Görtler mechanism develops a remarkable structure. In particular small wavelength vortices can be described in a strongly nonlinear state where they actually drive the mean flow. The breakdown of these vortices caused by a wavy vortex secondary instability can also be described asymptotically and the results obtained agree well with experimental observations. At order one vortex wavelengths the nonlinear nonparallel evolution of the vortices can only be described numerically. In this paper we have described a comparison of the results obtained using the spatial nonlinear approach of Hall(1988) and a temporal parallel flow simulation of Sabry and Liu(1988). The differences between the results which we have found suggest that temporal simulations do not qualitatively reproduce important features of the nonlinear stages of vortex growth.

Finally we close by making a few remarks about some recent work on vortex-wave interactions in boundary layer flows and channel flows. The point which we wish to make is that in parallel flows the interaction of oblique Tollmien-Schlichting waves can cause the generation of longitudinal vortex structures which have scales appropriate to Görtler vortices. The work of Hall and Smith (1988) shows that the linear vortex equations (1.4) have sufficient structure to describe oblique Tollmien-Schlichting waves so that an interaction between these waves and longitudinal vortex structures can be explained within the framework of the Görtler vortex equations. The interaction equations found by Hall and Smith have singular solutions of large amplitudes and these solutions can exist in the absence of wall curvature. Thus there can be a self-sustained interaction between longitudinal vortices and oblique waves. These results were developed further by Hall and Smith (1989a,b) who show that, in a flat plate boundary layer, small amplitude oblique

Tollmien-Schlichting waves occur as secondary bifurcations from two-dimensional waves. Subsequently these oblique waves drive a large amplitude vortex field of the type which is known to occur in some forms of boundary layer transition.

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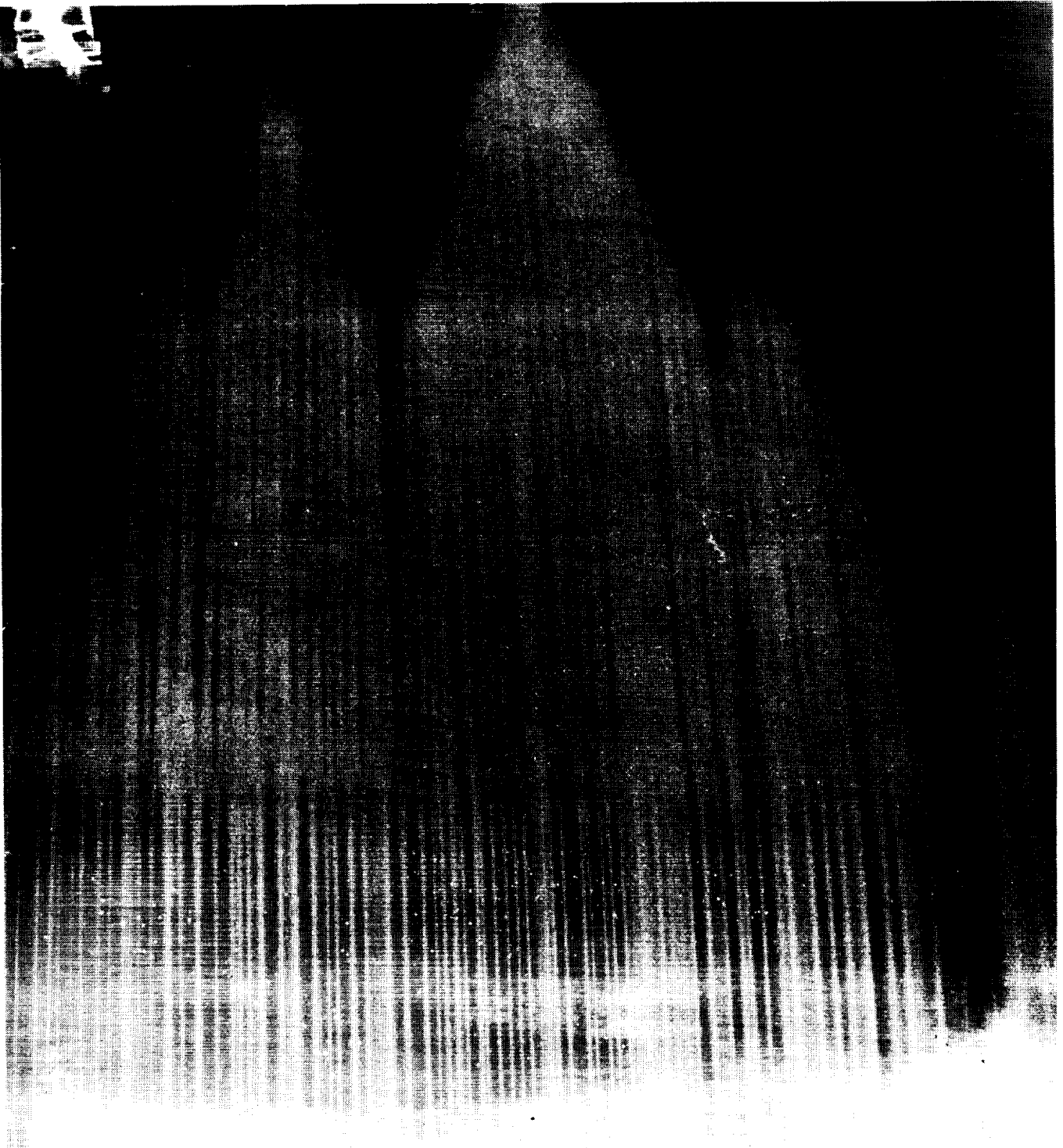
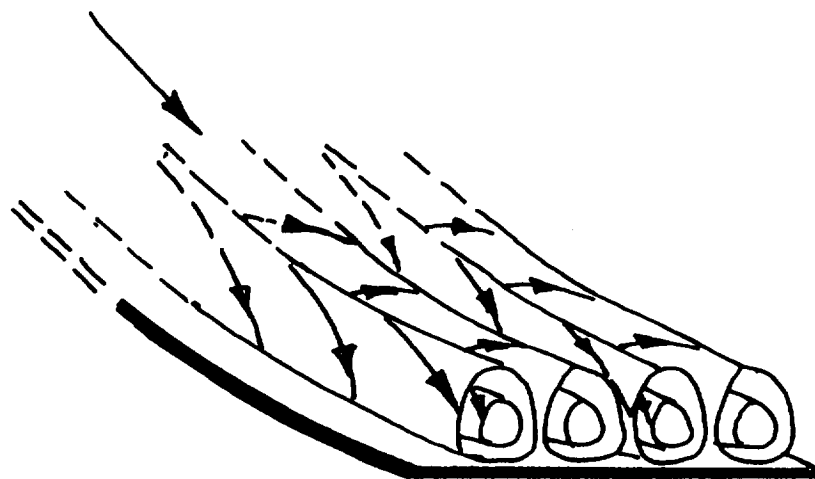


FIGURE 1.1 Görtler vortices in the concave section of a Laminar Flow Wing.





**FIGURE 1.2** The flow pattern associated with Görtler vortices.

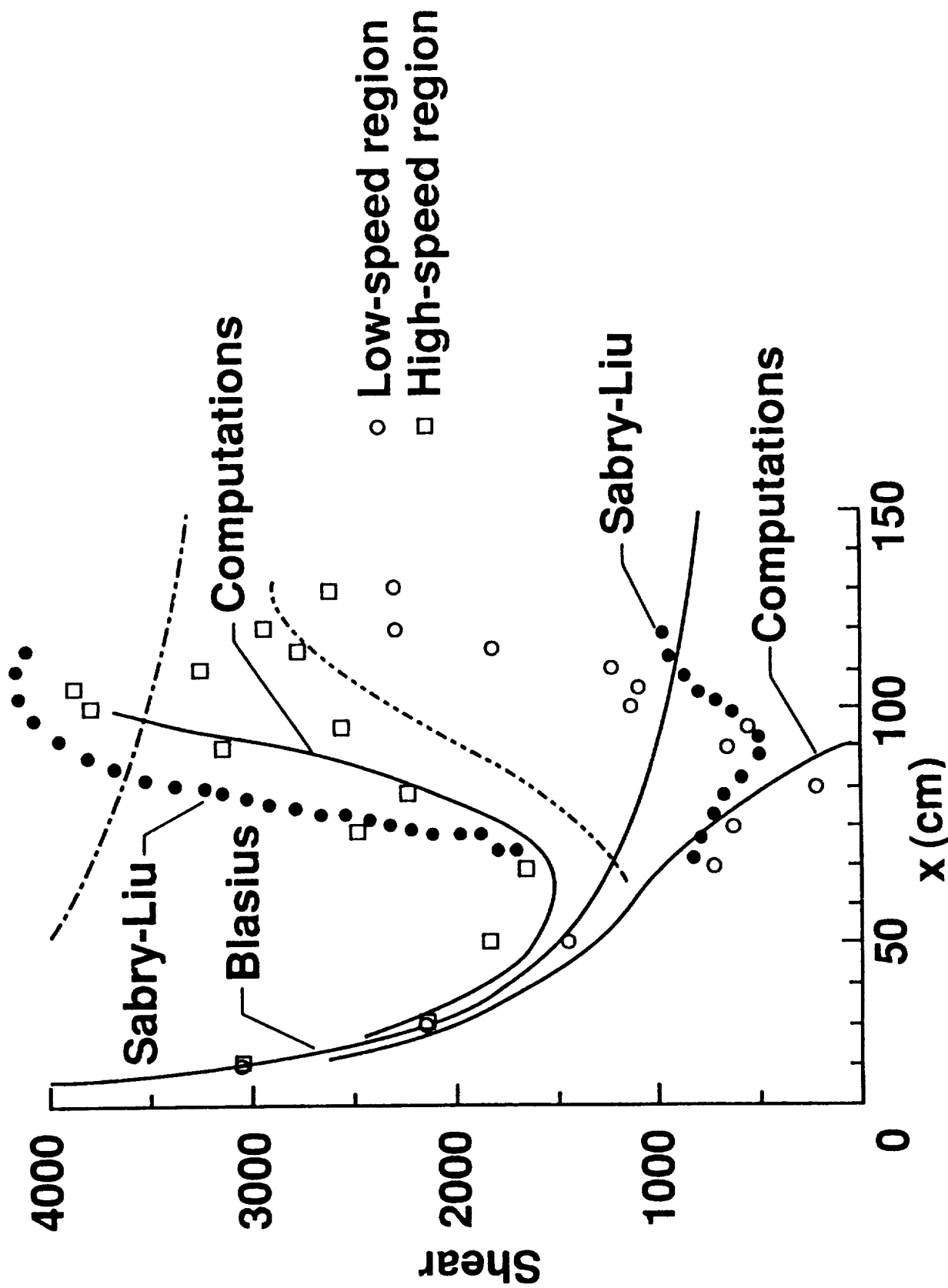
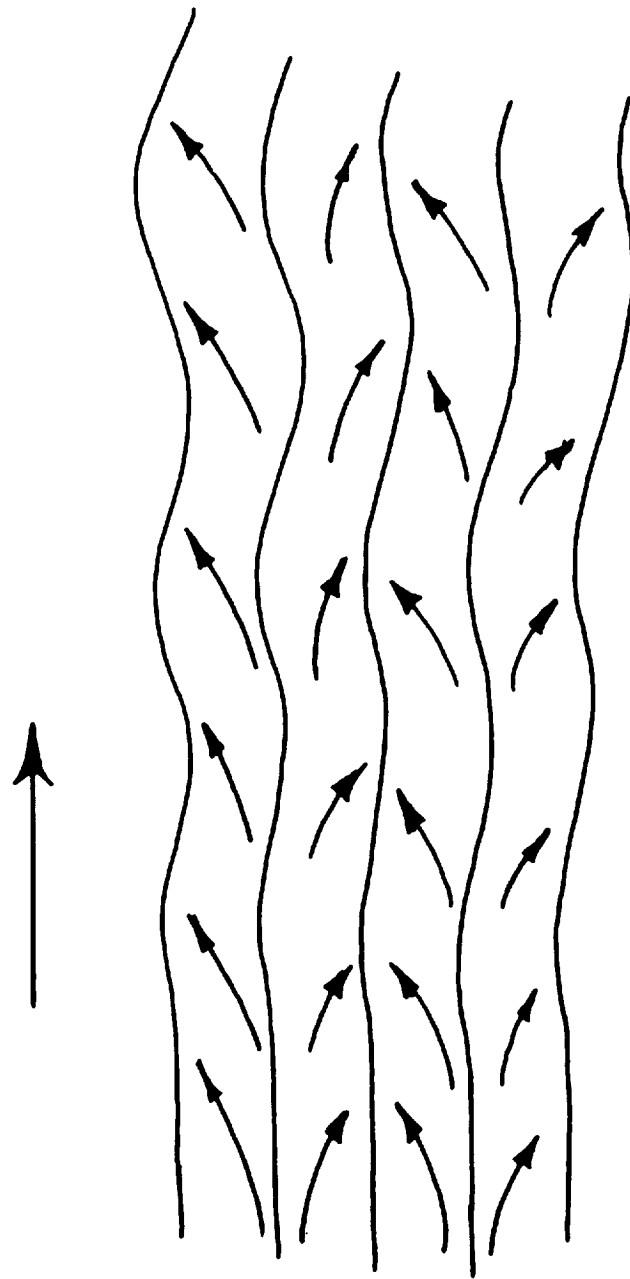


FIGURE 4.10 A comparison between the wall shear measured experimentally with that predicted by Sabry-Liu approach and the nonlinear spatial.



**FIGURE 4.8** The flow structure after the breakdown of the Hall-Lakin solution.

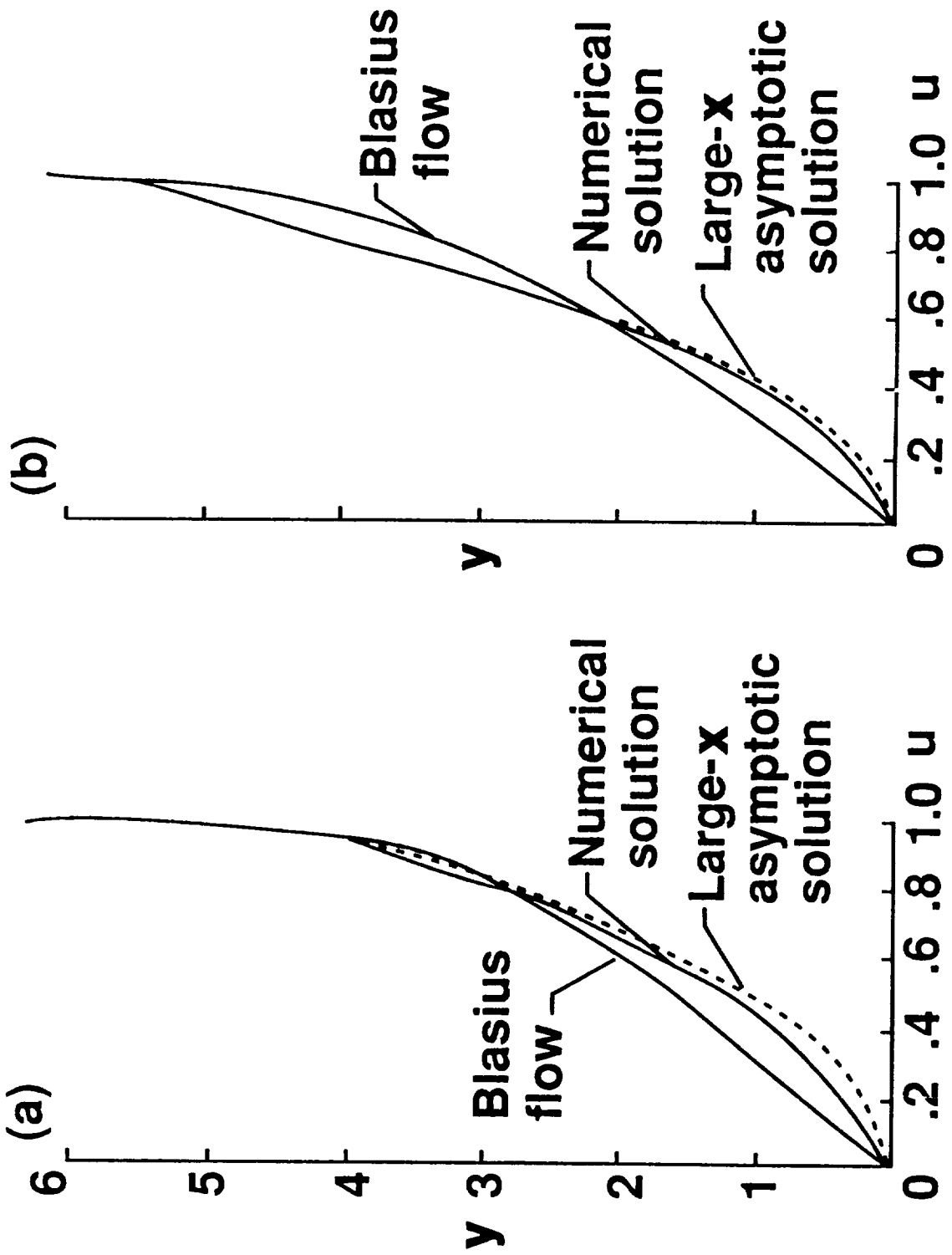
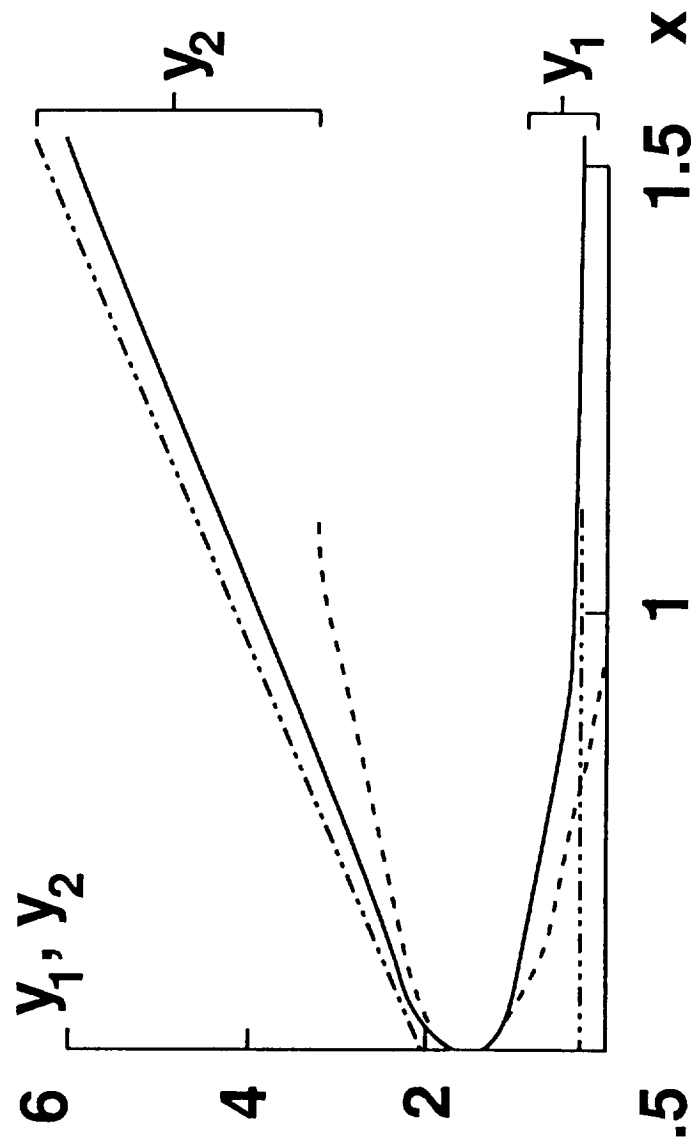


FIGURE 4.7 The total downstream velocity component corresponding to

$\chi = 2x, G_0 = 4.176$  and (a)  $x = 1.4$ , (b)  $x = 4.176$ .



**FIGURE 4.6** The locations of the shear layers  $y_1, y_2$  for the case  $\chi = 4.176$ . The continuous curve is the full numerical solution, --- is the weakly nonlinear solution, -·-·- the far downstream solution.

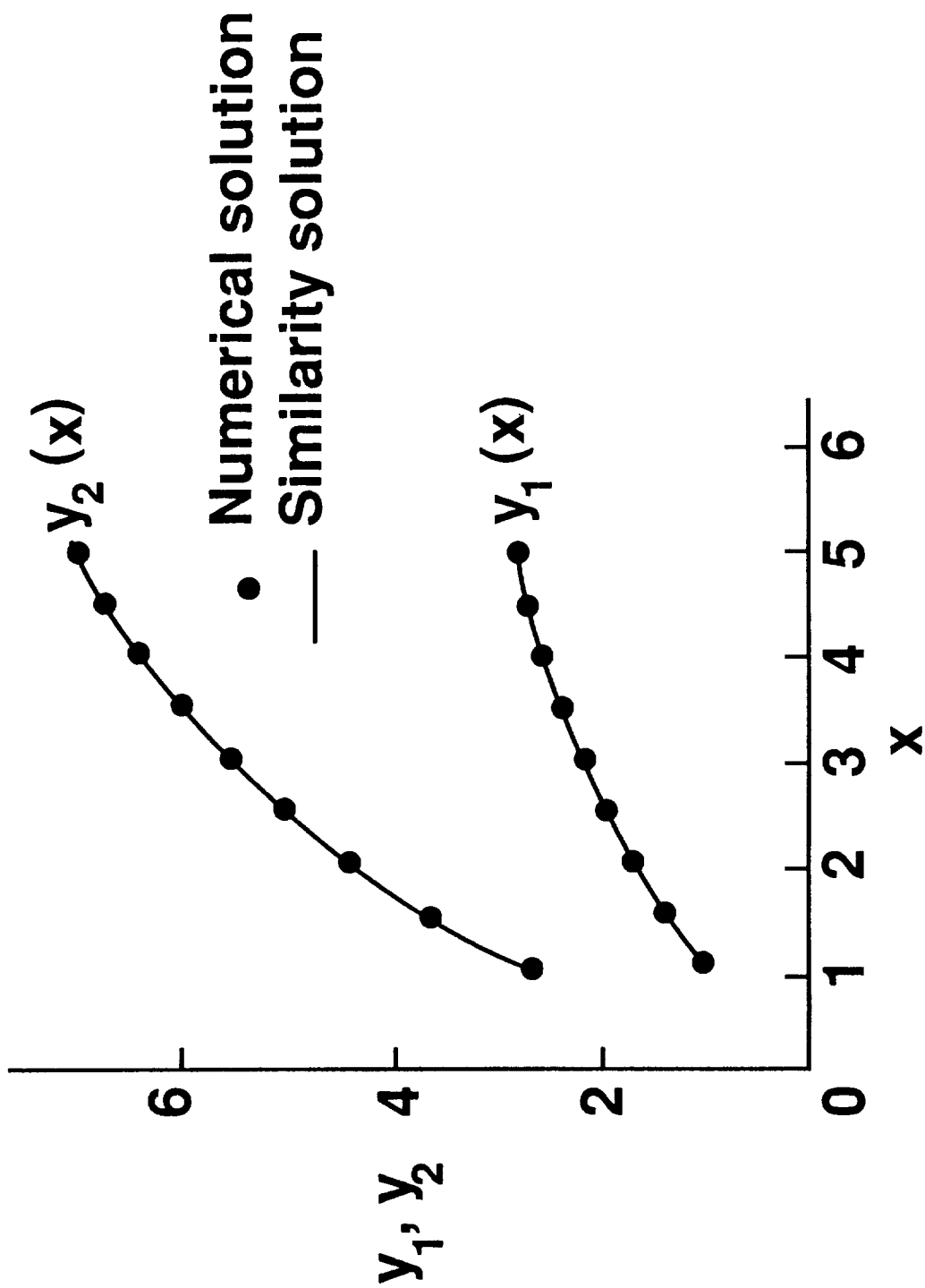


FIGURE 4.5 The locations of the shear layers  $y_1, y_2$ . for the case  $\chi \sim x^{\frac{1}{2}}$ .

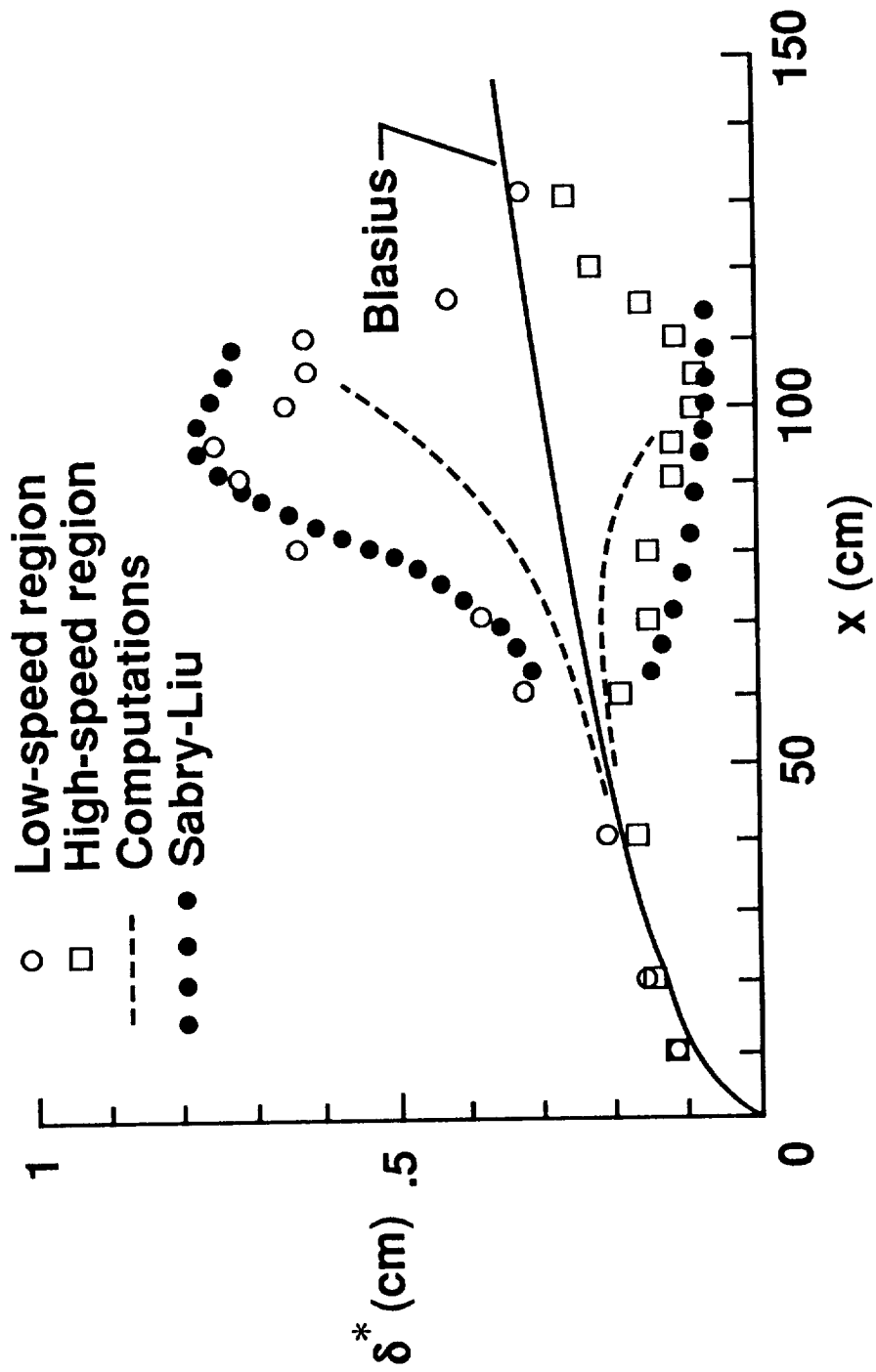


FIGURE 4.9 A comparison between the experimentally measured boundary layer thickness with that predicted by the Sabry-Liu approach together with nonlinear spatial results.

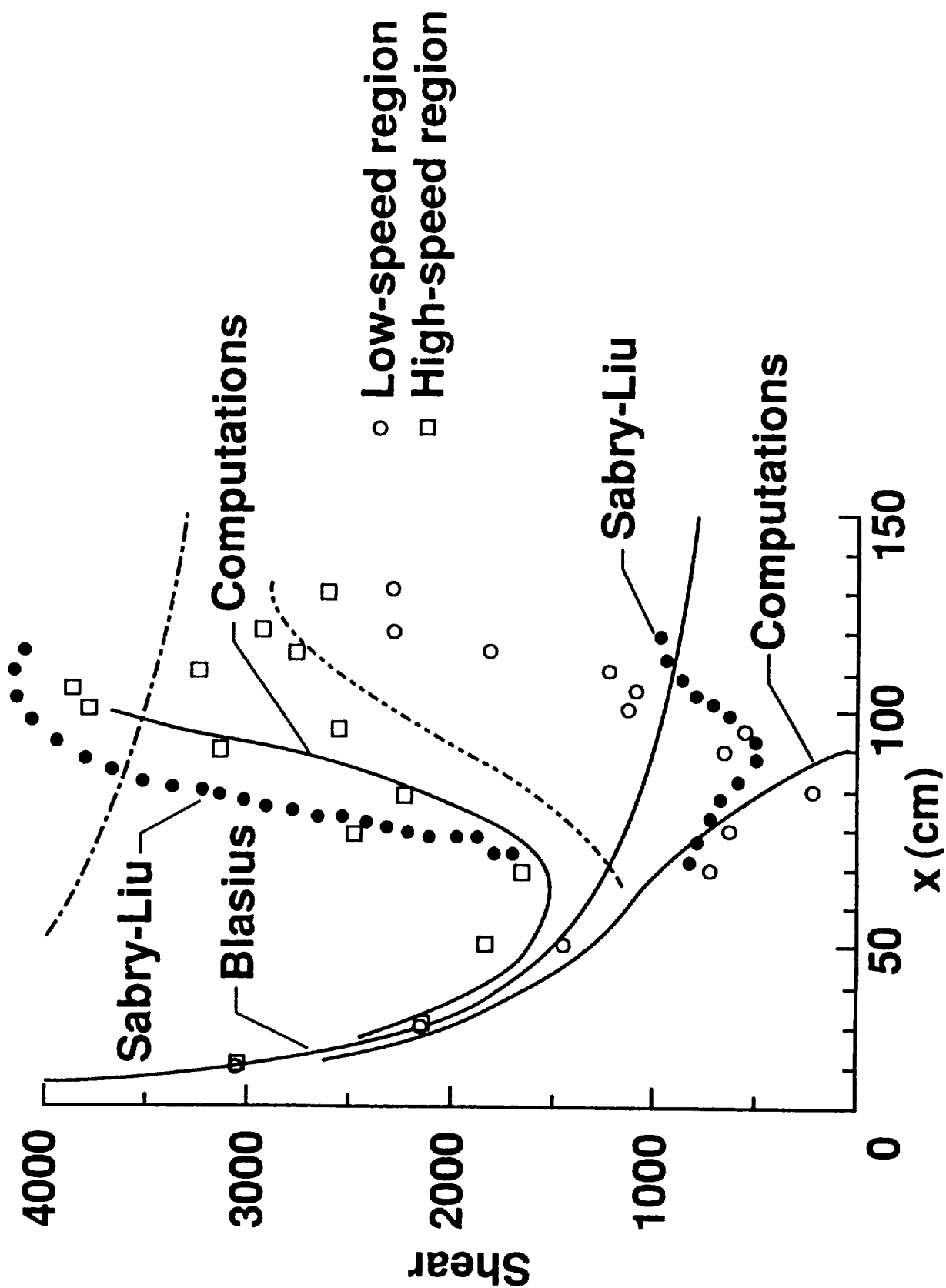


FIGURE 4.10 A comparison between the wall shear measured experimentally with that predicted by Sabry-Liu approach and the nonlinear spatial.





## Report Documentation Page

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16. Abstract Görtler vortices are thought to be the cause of transition in many fluid flows of practical importance. In this paper a review of the different stages of vortex growth is given. In the linear regime nonparallel effects completely govern this growth and parallel flow theories do not capture the essential features of the development of the vortices. A detailed comparison between the parallel and nonparallel theories is given and it is shown that at small vortex wavelengths the parallel flow theories have some validity; otherwise nonparallel effects are dominant. New results for the receptivity problem for Görtler vortices are given; in particular vortices induced by free-stream perturbations impinging on the leading edge of the wall are considered. It is found that the most dangerous mode of this type can be isolated and its neutral curve is determined. This curve agrees very closely with the available experimental data. A discussion of the different regimes of growth of nonlinear vortices is also given. Again it is shown that, unless the vortex wavelength is small, nonparallel effects are dominant. Some new results for nonlinear vortices of $O(1)$ wavelengths are given and compared to experimental observations. The agreement between theory and experiment is shown to be excellent up to the point where unsteady effects become important. For small wavelength vortices the nonlinear regime is of particular interest since there a strongly nonlinear theory can be developed.					
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